Further Differentiation and Applications

Prerequisites: Inverse function property; product, quotient and chain rules; inflexion points.

Maths Applications: Concavity; differentiability.

Real-World Applications: Particle motion; optimisation.

Derivative of Inverse Functions

Given a function f, the derivative of its inverse f^{-1} can be found.

Theorem:

Given a function f, the derivative of f^{-1} is given by,

$$D(f^{-1}) = \frac{1}{(Df) \circ f^{-1}}$$

In Leibniz notation, this formula takes on a much more memorable form, remembering that if $y = f(x)$ then $x = f^{-1}(y)$,

$$\frac{dx}{dy} = \frac{1}{\left(\frac{dy}{dx}\right)}$$

An example will be given (using both formulae) to show how the theorem is used.

Example 1

Obtain the derivative of the function defined by $y = f(x) = x^2$. Taking the positive root allows us to define an inverse function given by, $f^{-1}(x) = x^{1/2}$. Note that $Df = 2x$. Hence,

$$D(f^{-1}) = \frac{1}{2x^{1/2}}$$

Note that differentiating the inverse directly gives the same answer. Using the Leibniz formula instead, recalling that $x = f^{-1}(y) = y^{\frac{1}{2}}$,

$$\frac{dx}{dy} = \frac{d}{dy}\,x$$

$$\frac{dx}{dy} = \frac{d}{dy}\,y^{\frac{1}{2}}$$

$$\frac{dx}{dy} = \frac{1}{2y^{\frac{1}{2}}}$$

This is the same answer as above (the inverse function used here has variable y, so interchanging x and y gives the same result).

On the other hand,

$$\frac{1}{\left(\frac{dy}{dx}\right)} = \frac{1}{2x}$$

$$\frac{1}{\left(\frac{dy}{dx}\right)} = \frac{1}{2y^{\frac{1}{2}}}$$

Derivatives of Inverse Trigonometric Functions

The derivatives of arcsine, arccosine and arctangent can be found using the above formula.

- $\frac{d}{dx}\sin^{-1}x = \frac{1}{\sqrt{1 - x^2}}$
- $\frac{d}{dx}\cos^{-1}x = -\frac{1}{\sqrt{1 - x^2}}$
- $\frac{d}{dx}\tan^{-1}x = \frac{1}{1 + x^2}$

Example 2

Differentiate $y = \cos^{-1}(3x)$.

By the chain rule,

$$\frac{dy}{dx} = -\frac{1}{\sqrt{1 - (3x)^2}} \times \frac{d}{dx}(3x)$$

$$\frac{dy}{dx} = -\frac{3}{\sqrt{1 - 9x^2}}$$

Example 3

Differentiate $y = \sin^{-1}(x^3)$.

$$\frac{dy}{dx} = \frac{1}{\sqrt{1 - (x^3)^2}} \times \frac{d}{dx}(x^3)$$

$$\frac{dy}{dx} = \frac{3x^2}{\sqrt{1 - x^6}}$$

Example 4

Differentiate $y = \tan^{-1}\sqrt{2 + x}$.

$$\frac{dy}{dx} = \frac{1}{1 + (2 + x)} \times \frac{d}{dx}\sqrt{2 + x}$$

$$\frac{dy}{dx} = \frac{1}{3 + x} \times \frac{1}{2\sqrt{2 + x}}$$

$$\frac{dy}{dx} = \frac{1}{2(3 + x)\sqrt{2 + x}}$$

Often in this course, answers to derivatives will be freakishly complex looking, as in Example 4. Don't let this convince you that the answer is wrong. Just follow the rules and get an answer.

Implicit Differentiation

Implicit and Explicit Functions

Definition:

A function f is given **explicitly** (f is an **explicit function**) if the output value y is given in terms of the input value.

An explicit function is recognised when y is given as a function of x.

Definition:

A function f is given **implicitly** (f is an **implicit function**) if the output value y is not given in terms of the input value.

An implicit function is usually identified when the variables x and y are mixed up in a higgledy-piggledy manner.

An implicitly defined function may or may not be solved for y.

Example 5

$xy = 1$ defines y implicitly as a function of x. However, for $x \neq 0$, it can be written explicitly as $y = \frac{1}{x}$.

Example 6

The function $2y^2 - 3xy - 8\sin y + x^2 = 5$ defines y implicitly as a function of x; the expression on the LHS is so higgledy-piggledy in x and y, that y cannot be written explicitly as a function of x.

First Derivatives of Implicit Functions

To find $\frac{dy}{dx}$ for an implicitly defined function, differentiate both sides of the equality with respect to x and solve the resulting equation for $\frac{dy}{dx}$.

The chain rule will most likely be used and sometimes the product or quotient rule too. Just remember that y is a function of x.

Example 7

Find $\frac{dy}{dx}$ for the function defined implicitly by the equation $x^2 + xy^3 =$ 15.

First, write down what the intent is,

$$\frac{d}{dx}\left(x^2 + xy^3\right) = \frac{d}{dx}15$$

When there is a lonely constant on one side of the equation, it can be very easy to forget to differentiate it; so differentiate it first. Next, remember that y is a function of x, so the second term on the LHS needs an application of the product rule,

$$2x + \left(y^3 + 3xy^2\frac{dy}{dx}\right) = 0$$

Solving this equation for the derivative gives,

$$\frac{dy}{dx} = -\frac{\left(2x + y^3\right)}{3xy^2}$$

Example 8

Find $\frac{dy}{dx}$ when $6y^2 + 2xy - x^2 = 3$.

$$\frac{d}{dx}(6y^2 + 2xy - x^2) = \frac{d}{dx}3$$

$$12y\frac{dy}{dx} + 2\left(y + x\frac{dy}{dx}\right) - 2x = 0$$

$$(12y + 2x)\frac{dy}{dx} = 2x - 2y$$

$$\frac{dy}{dx} = \frac{2x - 2y}{12y + 2x}$$

$$\frac{dy}{dx} = \frac{x - y}{6y + x}$$

Functions defined implicitly by equations that look a little more menacing are still no match for implicit differentiation.

Example 9

Find $\frac{dy}{dx}$ when $2y^2 - 3xy - 8\sin y + x^2 = 4$.

$$\frac{d}{dx}(2y^2 - 3xy - 8\sin y + x^2) = \frac{d}{dx}4$$

$$4y\frac{dy}{dx} - 3\left(y + x\frac{dy}{dx}\right) - 8\cos y\frac{dy}{dx} + 2x = 0$$

$$(4y - 3x - 8\cos y)\frac{dy}{dx} = 3y - 2x$$

$$\frac{dy}{dx} = \frac{3y - 2x}{4y - 3x - 8\cos y}$$

Example 10

Find $\frac{dy}{dx}$ when $\sin x\ \ln(xy) + x^2 y = 7$.

$$\frac{d}{dx}(\sin x \ln(xy) + x^2 y) = \frac{d}{dx}7$$

$$\sin x\left(\frac{1}{y}\frac{dy}{dx}\right) + x^2\frac{dy}{dx} = -\cos x \ln(xy) - 2xy - \frac{1}{x}\sin x$$

$$x\sin x\frac{dy}{dx} + x^3 y\frac{dy}{dx} = -xy\cos x \ln(xy) - 2x^2y^2 - y\sin x$$

$$(x \sin x + x^3 y)\frac{dy}{dx} = -\, xy \cos x \ln (xy) - 2x^2 y^2 - y \sin x$$

$$\frac{dy}{dx} = -\frac{(xy \cos x \ln (xy) + 2x^2y^2 + y \sin x)}{x \sin x + x^3 y}$$

Second Derivatives of Implicit Functions

The second derivative can also be found using implicit differentiation. The idea is to differentiate a line of working that is used in finding the first derivative.

Example 11

Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ when $xy - x = 4$.

Differentiating implicitly we get,

$$y + x\frac{dy}{dx} - 1 = 0$$

From this we get,

$$\frac{dy}{dx} = \frac{1 - y}{x}$$

Differentiate implicitly the line of working to get,

$$\frac{dy}{dx} + \left(\frac{dy}{dx} + x\frac{d^2y}{dx^2}\right) = 0$$

$$x\frac{d^2y}{dx^2} = -2\frac{dy}{dx}$$

$$x\frac{d^2y}{dx^2} = -2\left(\frac{1 - y}{x}\right)$$

$$\frac{d^2y}{dx^2} = \frac{2(y - 1)}{x^2}$$

Logarithmic Differentiation

Another technique to add to our arsenal of differentiating functions involves taking natural logarithms of both sides of an equation before differentiating. This technique is known as **logarithmic differentiation**. Logarithmic differentiation should be used when any one of the following indicators are present.

- Bracketed terms with fractional powers.
- Variable is in the power.
- Product or quotient of more than 2 functions.

Example 12

Differentiate $y = \dfrac{(3x - 2)^{2/3}(1 - 2x)^{3/2}}{(4x + 7)^{3/4}}$.

Taking natural logarithms of both sides and using the ' power comes down ' rule for logarithms gives,

$$\ln y = \frac{2}{3}\ln(3x - 2) + \frac{3}{2}\ln(1 - 2x) - \frac{3}{4}\ln(4x + 7)$$

This is much friendlier to differentiate. Differentiating and simplifying each term on the RHS gives,

$$\frac{1}{y}\frac{dy}{dx} = \frac{2}{3x - 2} - \frac{3}{1 - 2x} - \frac{3}{4x + 7}$$

$$\frac{dy}{dx} = y\left(\frac{2}{3x - 2} - \frac{3}{1 - 2x} - \frac{3}{4x + 7}\right)$$

Remembering what y actually is gives,

$$\frac{dy}{dx} = \frac{(3x - 2)^{2/3}(1 - 2x)^{3/2}}{(4x + 7)^{3/4}} \left(\frac{2}{3x - 2} - \frac{3}{1 - 2x} - \frac{3}{4x + 7} \right)$$

<u>Example 13</u>

Differentiate $y = x\, e^{-3x} \cos x$.

Taking natural logarithms gives,

$$\ln y = \ln x + \ln (e^{-3x}) + \ln (\cos x)$$

$$\ln y = \ln x - 3x + \ln (\cos x)$$

Differentiating gives (notice that term on the LHS cropping up again),

$$\frac{1}{y}\frac{dy}{dx} = \frac{1}{x} - 3 + \frac{(-\sin x)}{\cos x}$$

$$\frac{dy}{dx} = y\left(\frac{1}{x} - 3 - \tan x \right)$$

$$\frac{dy}{dx} = x\, e^{-3x} \cos x \left(\frac{1}{x} - 3 - \tan x \right)$$

$$\frac{dy}{dx} = e^{-3x} \cos x - 3x\, e^{-3x} \cos x - x\, e^{-3x} \sin x$$

<u>Example 14</u>

Differentiate $y = x^{\sin x}$.

Taking natural logarithms gives,

$$\ln y = \sin x \,.\, \ln x$$

Differentiating (using the product rule) gives,

$$\frac{1}{y}\frac{dy}{dx} = \cos x \,.\, \ln x + \sin x \,.\, \frac{1}{x}$$

$$\frac{dy}{dx} = y(\cos x \,.\, \ln x + x^{-1}\sin x)$$

$$\frac{dy}{dx} = x^{\sin x}(\cos x \,.\, \ln x + x^{-1}\sin x)$$

Parametric Differentiation

It is possible to write a function $y = f(x)$ differently as 2 functions of a common variable. This sometimes makes it easier to work out $\frac{dy}{dx}$.

Parametric Functions

Definition:

A curve is defined **parametrically** if it can be described by functions x and y, called **parametric functions** (aka **parametric equations of the curve**), of a common variable known as the **parameter**.

Normally, the parameter is denoted by t or θ, and the parametric functions x and y are thus written as $x(t)$ and $y(t)$ or $x(\theta)$ and $y(\theta)$. It often helps to think of the parameter as time; as the value of t changes, a curve is generated in the $x - y$ plane. Think of an ant walking about in the $x - y$ plane; for each value of t, it has a certain position (i.e. coordinate) in the plane given by $(x(t), y(t))$. Sometimes the curve is a function, but it can be any type of ' shape ' in the plane (for example, a circle). Many ' shapes ' are often easier to describe implicitly.

Example 15

Show that the point $(0, -1)$ lies on the curve described by the parametric equations $x = \cos\theta$, $y = \sin\theta$ ($\theta \in [0, 2\pi]$), stating the value of θ.

Note that $x^2 + y^2 = 1$, i.e. the parametric equations define a circle with centre $(0, 0)$ and radius 1. The point $(0, -1)$ clearly lies on this circle and the angle is obviously $3\pi/2$. Another way to see this is to put $x = 0$ and $y = -1$ into the parametric equations and show that there is only one value of θ that satisfies both equations. Indeed, $0 = \cos\theta$ leads to

$\theta = \pi/2$ or $3\pi/2$, whereas $-1 = \sin\theta$ leads to $\theta = 3\pi/2$. So, there is a common value that satisfies both equations and it is $\theta = 3\pi/2$.

Example 16

Show that the point (2, 5) does not lie on the curve defined parametrically by $x = 2t$, $y = 4t^2$.

If the given point did lie on the curve, then $2 = 2t \Rightarrow t = 1$. However, $t = 1$ gives $y = 4$, not 5. Thus, (2, 5) does not lie on the curve.

First Derivatives of Parametric Functions

Theorem:

Given 2 parametric functions $x(t)$ and $y(t)$, the derivative of y with respect to x is given by,

$$\frac{dy}{dx} = \frac{dy}{dt} \div \frac{dx}{dt}$$

In Newton Notation, this is written as,

$$\frac{dy}{dx} = \frac{\dot{y}}{\dot{x}}$$

Example 17

Find $\frac{dy}{dx}$ when $x = 2t^2$, $y = \ln t$.

The derivatives with respect to t are $\dot{x} = 4t$ and $\dot{y} = \frac{1}{t}$. Hence,

$$\frac{dy}{dx} = \frac{\dot{y}}{\dot{x}}$$

$$\frac{dy}{dx} = \frac{1/t}{4t}$$

$$\frac{dy}{dx} = \frac{1}{4t^2}$$

Example 18

Find the equation of the tangent line to the curve at $t = 2$ defined parametrically by the equations $x = \frac{t}{1 - t^2}$, $y = \frac{1 + t^2}{1 - t^2}$.

The derivatives are,

$$\frac{dx}{dt} = \frac{1 + t^2}{(1 - t^2)^2} \quad , \quad \frac{dy}{dt} = \frac{4t(1 - t^2)}{(1 - t^2)^2}$$

Hence,

$$\frac{dy}{dx} = \frac{dy}{dt} \div \frac{dx}{dt}$$

$$\frac{dy}{dx} = \frac{4t(1 - t^2)}{1 + t^2}$$

At $t = 2$, $x = -\frac{2}{3}$, $y = -\frac{5}{3}$ and $\frac{dy}{dx} = -\frac{24}{5}$. Hence, the equation of the tangent line is,

$$y + \frac{5}{3} = -\frac{24}{5}\left(x + \frac{2}{3}\right)$$

After a wee bit of simplifying, this becomes,

$$72x + 15y + 73 = 0$$

Second Derivatives of Parametric Functions

Theorem:

Given 2 parametric functions $x\,(t)$ and $y\,(t)$, the second derivative of y with respect to x is given by,

$$\frac{d^2y}{dx^2} = \frac{d}{dt}\left(\frac{\dot{y}}{\dot{x}}\right) \times \frac{1}{\dot{x}}$$

Evaluating this leads to,

$$\frac{d^2y}{dx^2} = \frac{\dot{x}\,\ddot{y} - \dot{y}\,\ddot{x}}{\dot{x}^3}$$

Example 19

Find $\frac{d^2y}{dx^2}$ when $x = 3 + 3t$ and $y = 4 - 4t^2$.

If using the second form, calculate the relevant quantities first. So, $\dot{x} = 3$, $\ddot{x} = 0$, $\dot{y} = -8t$ and $\ddot{y} = -8$. Then use the formula,

$$\frac{d^2y}{dx^2} = \frac{\dot{x}\,\ddot{y} - \dot{y}\,\ddot{x}}{\dot{x}^3}$$

$$\frac{d^2y}{dx^2} = \frac{3.(-8) - (-8t).0}{3^3}$$

$$\frac{d^2y}{dx^2} = -\frac{24}{27}$$

$$\frac{d^2y}{dx^2} = -\frac{8}{9}$$

Example 20

Show that the curve defined parametrically by the equations $x = t - \frac{1}{t^2}$ and $y = t + \frac{1}{t^2}$ has no point of inflexion.

First note that $t \neq 0$. If there is a P of I, then either $\frac{d^2y}{dx^2}$ does not exist or it is obtained by solving $\frac{d^2y}{dx^2} = 0$. To obtain the second derivative, use the first form (it's easier) to get,

$$\frac{d^2y}{dx^2} = \frac{4t^6}{(t^4 + 1)^3}$$

Solving this for t gives $t = 0$. But this contradicts the fact that $t \neq 0$. Alternatively, since $t \neq 0$, the equation $\frac{d^2y}{dx^2} = 0$ has no solutions for t. Either way, there is no P of I.

Planar Motion

Motion in a plane is often best described by parametric equations. The following definitions extend the 1D definitions from Unit 1 to 2D.

Definition:

Planar motion is motion in a plane and is described by 2 functions of time $x(t)$ and $y(t)$.

Definition:

The displacement of a particle at time t in a plane is described by the **displacement vector**,

$$\mathbf{s}(t) \overset{def}{=} (x(t), y(t)) = x(t)\,\mathbf{i} + y(t)\,\mathbf{j}$$

The **magnitude of displacement**, aka **distance (from the origin)**, at time t is,

$$|\mathbf{s}(t)| \overset{def}{=} \sqrt{x^2 + y^2}$$

The **direction measured from the x – axis** (aka **direction of displacement**), θ, at any instant of time t is,

$$\tan \theta = \frac{y}{x}$$

where θ is the angle between $x\,\mathbf{i}$ and $\boldsymbol{s}$.

<u>Definition:</u>

The velocity of a particle at time t in a plane is described by the **velocity vector**,

$$\boldsymbol{v}(t) \overset{def}{=} \frac{d\boldsymbol{s}}{dt} = (\dot{x}(t), \dot{y}(t)) = \dot{x}(t)\,\mathbf{i} + \dot{y}(t)\,\mathbf{j}$$

The **magnitude of velocity**, aka **speed**, at time t, is,

$$|\boldsymbol{v}(t)| \overset{def}{=} \sqrt{\dot{x}^2 + \dot{y}^2}$$

The **direction of motion** (aka **direction of velocity**), η, at any instant of time t is,

$$\tan \eta = \frac{\dot{y}}{\dot{x}}$$

where η is the angle between $\dot{x}\,\mathbf{i}$ and $\boldsymbol{v}$.

<u>Definition:</u>

The acceleration of a particle at time t in a plane is described by the **acceleration vector**,

$$\boldsymbol{a}(t) \overset{def}{=} \frac{d\boldsymbol{v}}{dt} = (\ddot{x}(t), \ddot{y}(t)) = \ddot{x}(t)\,\mathbf{i} + \ddot{y}(t)\,\mathbf{j}$$

The **magnitude of acceleration**, at time t, is,

$$|\boldsymbol{a}(t)| \overset{def}{=} \sqrt{\ddot{x}^2 + \ddot{y}^2}$$

The **direction of acceleration**, ψ, at any instant of time t, is,

$$\tan \psi = \frac{\ddot{y}}{\ddot{x}}$$

where ψ is the angle between $\ddot{x}$ **i** and ***a***.

Example 21

Find the magnitude and direction of the displacement, velocity and acceleration at time $t = 2$ of the particle whose equations of motion are $x = t^3 - 2t$ and $y = t^2 + 4t$.

We deal first with the displacement. $x(2) = 4$ and $y(2) = 12$. Hence, the magnitude of displacement is $\sqrt{16 + 144} = \sqrt{160} = 4\sqrt{10}$ metres. The direction is found by solving $\tan \theta = 3$, which gives $71 \cdot 6\,°$.

The velocity components are given by $\dot{x} = 3t^2 - 2$ and $\dot{y} = 2t + 4$. So, $\dot{x}(2) = 10$ and $\dot{y}(2) = 8$. Hence, the magnitude of velocity is $\sqrt{100 + 64} = \sqrt{164} = 2\sqrt{41}$ metres per second. The direction is found by solving $\tan \eta = 4/5$, which gives $38 \cdot 7\,°$.

The acceleration components are given by $\ddot{x} = 6t$ and $\ddot{y} = 2$. So, $\ddot{x}(2) = 12$ and $\ddot{y}(2) = 2$. Hence, the magnitude of acceleration is $\sqrt{144 + 4} = \sqrt{148} = 2\sqrt{37}$ metres per second squared. The direction is found by solving $\tan \psi = 1/6$, which gives $9 \cdot 5\,°$.

Related Rates of Change

Definition:

A **related rate of change** refers to an equation involving derivatives, in particular, derivatives arising from y as a function of x and both x and y as functions of a third variable u.

Problems involving related rates of change arise naturally in real-life. To solve them requires use of the Chain Rule and occasionally the Leibniz relation $\dfrac{dx}{dy} = \dfrac{1}{\left(\dfrac{dy}{dx}\right)}$.

Example 22

If a spherical balloon is inflated at a constant rate of 240 cubic centimetres per second, find the rate at which the radius is increasing when the diameter is 16 cm.

Something about a sphere, it's diameter (and hence radius), rate of change of volume with respect to time and rate of change of radius with respect to time. Sounds like the volume of a sphere would be a good start.

$$V = \frac{4}{3}\pi r^3$$

Remember that V and r are secretly functions of time, t. Differentiate both sides w.r.t. t to get,

$$\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$$

Rearranging this gives,

$$\frac{dr}{dt} = \frac{1}{4\pi r^2}\frac{dV}{dt}$$

If the diameter is 16 cm, then the radius is 8 cm. Hence,

$$\frac{dr}{dt} = \frac{1}{4\pi(64)} . 240$$

$$\frac{dr}{dt} = \frac{240}{256\pi}$$

$$\frac{dr}{dt} = \frac{15}{16\pi} \text{ cm/s}$$

Example 23

In a capacitive circuit, the formula for the total capacitance C of 2 capacitors in series is,

$$\frac{1}{C} = \frac{1}{C_1} + \frac{1}{C_2}$$

If C_1 is increasing at a rate of $0 \cdot 5$ farads per minute and C_2 is decreasing at a rate of $0 \cdot 9$ farads per minute, at what rate is C changing (to 1 d. p.) when $C_1 = 2$ farads and $C_2 = 1$ farad ?

Remember that C, C_1 and C_2 are secretly functions of time. Firstly, when $C_1 = 2$ farads and $C_2 = 1$ farad, $C = 2/3$ farad (easy adding fractions; don't forget to reciprocalise !). Next, differentiate the given equation w.r.t. t to get,

$$-\frac{1}{C^2}\frac{dC}{dt} = -\frac{1}{C_1^2}\frac{dC_1}{dt} - \frac{1}{C_2^2}\frac{dC_2}{dt}$$

With a little algebraic jiggery-pokery, this becomes,

$$\frac{dC}{dt} = \left(\frac{C}{C_1}\right)^2\frac{dC_1}{dt} + \left(\frac{C}{C_2}\right)^2\frac{dC_2}{dt}$$

List the quantities and their values. $C = 2/3$, $C_1 = 2$, $C_2 = 1$, $\frac{dC_1}{dt} = 0 \cdot 5$ and $\frac{dC_2}{dt} = -0 \cdot 9$ (it's negative because it's decreasing). Now it's just number crunching,

$$\frac{dC}{dt} = \left(\frac{1}{3}\right)^2 . (0 \cdot 5) + \left(\frac{2}{3}\right)^2 . (-0 \cdot 9)$$

$$\frac{dC}{dt} = \frac{1}{18} - \frac{2}{5}$$

$$\frac{dC}{dt} = -0 \cdot 3 \text{ farads (1 d. p.)}$$