# Further Integration

Prerequisites: Integration by substitution; standard integrals; completing the square; partial fractions.

Maths Applications: Solving differential equations.

Real-World Applications: Population growth.

## Integrals of Inverse Trigonometric Functions

From the derivatives of the inverse trigonometric functions, we get some more integrals for free.

- $\displaystyle\int \frac{dx}{\sqrt{1 - x^2}} = \sin^{-1} x + C$
- $\displaystyle\int -\frac{dx}{\sqrt{1 - x^2}} = \cos^{-1} x + C$
- $\displaystyle\int \frac{dx}{1 + x^2} = \tan^{-1} x + C$

Clearly, the first 2 integrals are very closely related. Using substitutions, the above 3 integrals have general counterparts,

- $\displaystyle\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1}\left(\frac{x}{a}\right) + C$
- $\displaystyle\int -\frac{dx}{\sqrt{a^2 - x^2}} = \cos^{-1}\left(\frac{x}{a}\right) + C$
- $\displaystyle\int \frac{dx}{a^2 + x^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C$

The first and last of these tend to arise the most often.

Example 1

Integrate $\dfrac{1}{\sqrt{9 - x^2}}$.

$$\int \frac{dx}{\sqrt{9 - x^2}} = \sin^{-1}\left(\frac{x}{3}\right) + C$$

Example 2

Integrate $\dfrac{1}{4 + 7x^2}$.

$$\int \frac{dx}{4 + 7x^2} = \int \frac{dx}{7(4/7 + x^2)}$$

$$= \frac{1}{7}\int \frac{dx}{\left(2/\sqrt{7}\right)^2 + x^2}$$

$$= \frac{1}{2\sqrt{7}}\tan^{-1}\left(\frac{x\sqrt{7}}{2}\right) + C$$

## Integrals of Rational Functions

Rational functions can be classed into many types, but the ones we will study in this course generally involve polynomials of fairly low degrees.

We will use 3 general types of integrals in what follows.

- $\displaystyle\int \frac{1}{px + q}\, dx = \frac{1}{p}\ln|px + q| + C$
- $\displaystyle\int \frac{Df}{f}\, dx = \ln|f| + C$
- $\displaystyle\int \frac{dx}{a^2 + x^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C$

## Constant Divided by a Linear

Example 3

Find $\displaystyle\int \frac{4\ dx}{3 - 11x}$.

This type of integral will always give some kind of natural logarithm. To work out the integral, make the substitution $u = 3 - 11x$. Then $du = -11\ dx$. Hence,

$$\begin{aligned}\int \frac{4\ dx}{3 - 11x} &= -\frac{4}{11}\int \frac{du}{u} \\ &= -\frac{4}{11}\ln|u| + C \\ &= -\frac{4}{11}\ln|3 - 11x| + C\end{aligned}$$

After a little while, you should be able to do these in your head.

## Constant Divided by a Linear Squared

Example 4

Find $\displaystyle\int \frac{3\ dx}{(5 + 4x)^2}$.

This is a Higher integral. To do it using substitution, let $u = 5 + 4x$. Then $du = 4\ dx$. Hence,

$$\begin{aligned}\int \frac{3\ dx}{(5 + 4x)^2} &= \frac{3}{4}\int u^{-2}\ du \\ &= -\frac{3}{4}u^{-1} + C \\ &= -\frac{3}{4(5 + 4x)} + C\end{aligned}$$

## Constant Divided by an Irreducible Quadratic

Compare the following example with Example 8, which contains a quadratic that is factorisable.

Example 5

Find $\int \frac{7\ dx}{3x^2 + 4}$.

First take out the 7 and then factorise out a 3 from the denominator to get,

$$\int \frac{7\ dx}{3x^2 + 4} = 7 \int \frac{dx}{3(x^2 + 4/3)}$$

$$= \frac{7}{3} \int \frac{dx}{(x^2 + 4/3)}$$

This smells suspiciously like inverse tangent. Hence,

$$\int \frac{7\ dx}{3x^2 + 4} = \frac{7}{3} \int \frac{dx}{\left(2/\sqrt{3}\right)^2 + x^2}$$

$$= \frac{7}{2\sqrt{3}} \tan^{-1}\left(\frac{x\sqrt{3}}{2}\right) + C$$

Example 6

Find $\int \frac{dx}{x^2 + 4x + 7}$.

This involves completing the square first,

$$\int \frac{dx}{x^2 + 4x + 7} = \int \frac{dx}{(x + 2)^2 + 3}$$

Making the substitution $u = x + 2$ gives (check !),

$$\int \frac{dx}{x^2 + 4x + 7} = \frac{1}{\sqrt{3}} \tan^{-1}\left(\frac{x + 2}{\sqrt{3}}\right) + C$$

## Simple Linear Divided by an Irreducible Symmetric Quadratic

By a ' simple linear ' term, we mean a linear term without the constant term. By an ' irreducible symmetric quadratic ', we mean an irreducible quadratic without the $x$ term.

Example 7

Find $\int \frac{5x\ dx}{3x^2 + 4}$.

These types can always be done by making a substitution (always the denominator). So, let $u = 3x^2 + 4$. Then $du = 6x\,dx$. Hence,

$$\int \frac{5x\ dx}{3x^2 + 4} = \frac{5}{6}\int \frac{du}{u}$$

$$= \frac{5}{6} \ln |u| + C$$

$$= \frac{5}{6} \ln \left|3x^2 + 4\right| + C$$

## General Linear Divided by an Irreducible Quadratic

This type is a combination of the above two.

Example 8

Find $\int \frac{5x + 7\ dx}{3x^2 + 4}$.

This can be split up as,

$$\int \frac{5x + 7\ dx}{3x^2 + 4} = \int \frac{5x\ dx}{3x^2 + 4} + \int \frac{7\ dx}{3x^2 + 4}$$

$$= \frac{5}{6} \ln \left| 3x^2 + 4 \right| + \frac{7}{2\sqrt{3}} \tan^{-1}\left( \frac{x\sqrt{3}}{2} \right) + C$$

using Examples 5 and 7.

## *Integration by Partial Fractions*

Rational functions involving the use of partial fractions can be integrated using the above examples and the following reminders as a guide (the arrow means ' integrates to give ').

- $\dfrac{1}{px + q} \rightarrow \dfrac{1}{p} \ln \left| px + q \right| + C$
- $\dfrac{1}{(px + q)^2} \rightarrow -\dfrac{1}{p} \dfrac{1}{px + q} + C$
- $\dfrac{1}{(x + a)^2 + b^2} \rightarrow \dfrac{1}{b} \tan^{-1}\left( \dfrac{x + a}{b} \right) + C$

In the last case above, the quadratic is irreducible.

<u>Example 9</u>

Find $\displaystyle\int \frac{x\ dx}{x^2 - 12x + 36}$.

Factorising the denominator and using partial fractions gives,

$$\int \frac{x\ dx}{x^2 - 12x + 36} = \int \frac{x\ dx}{(x - 6)^2}$$

$$= \int \left( \frac{1}{x - 6} + \frac{6}{(x - 6)^2} \right) dx$$

Splitting up the integral and integrating each bit gives,

$$\int \frac{x\,dx}{x^2 - 12x + 36} = \ln|x - 6| - \frac{6}{x - 6} + C$$

Example 10

Evaluate $\displaystyle\int_0^2 \frac{4}{(x^2 + 2x + 5)(x + 1)}\,dx$.

Performing partial fractions (check !) on the integrand results in,

$$\int_0^2 \frac{4}{(x + 1)(x^2 + 2x + 5)}\,dx = \int_0^2 \left(\frac{1}{(x + 1)} - \frac{(x + 1)}{(x^2 + 2x + 5)}\right) dx$$

$$= \left[\ln(x + 1) - \tfrac{1}{2}\ln(x^2 + 2x + 5)\right]_0^2$$

$$= \left[\ln\left(\frac{x + 1}{\sqrt{x^2 + 2x + 5}}\right)\right]_0^2$$

$$= \ln\left(\frac{3}{\sqrt{13}}\right) - \ln\left(\frac{1}{\sqrt{5}}\right)$$

$$= \ln\left(\frac{3\sqrt{5}}{\sqrt{13}}\right)$$

Example 11

Evaluate $\displaystyle\int_2^{2\sqrt{3}} \frac{x - 7}{x^2 + 4}\,dx$.

First split up the integral as,

$$\int_2^{2\sqrt{3}} \frac{x - 7}{x^2 + 4}\,dx = \int_2^{2\sqrt{3}} \frac{x}{x^2 + 4}\,dx - \int_2^{2\sqrt{3}} \frac{7}{x^2 + 4}\,dx$$

The first integral will give a natural logarithm (use a simple substitution if you can't see this straight away) while the second will give an inverse tangent,

$$\int_2^{2\sqrt{3}} \frac{x - 7}{x^2 + 4}\, dx = \left[\tfrac{1}{2} \ln (x^2 + 4) - \tfrac{7}{2} \tan^{-1} (x/2)\right]_2^{2\sqrt{3}}$$

$$= \left(\tfrac{1}{2} \ln 16 - \tfrac{7}{2} \tan^{-1} \sqrt{3}\right) - \left(\tfrac{1}{2} \ln 8 - \tfrac{7}{2} \tan^{-1} 1\right)$$

Collecting the logarithm terms together and remembering exact values for tangent gives,

$$= \ln\left(\frac{\sqrt{16}}{\sqrt{8}}\right) - \frac{7}{2}\left(\frac{\pi}{3}\right) + \frac{7}{2}\left(\frac{\pi}{4}\right)$$

$$= \ln \sqrt{2} - \frac{7\pi}{24}$$

Always leave answers as exact values, unless told do to otherwise. Remember that definite integrals can be approximated very accurately using a graphics calculator; use this to check if your answer is reasonable.

## Integration by Parts

A product of functions can sometimes be integrated using the following.

Theorem (**Integration by Parts**):

A product of 2 functions of the variable $x$ can be integrated (indefinitely) by using the **Integration by Parts Formula**,

$$\int u\,(D\,v)\; dx = u\,v - \int (D\,u)\,v\; dx$$

The formula for integrating by parts in definite form is,

$$\int_a^b u\,(D\,v)\; dx = \left[u\,v\right]_a^b - \int_a^b (D\,u)\,v\; dx$$

Notice that one function, $u$, is to be differentiated while the other function, $Dv$, is to be integrated. How do we decide which one to differentiate ? The general rule of thumb is, ' to make the integral on the RHS of the Integration by Parts Formula simpler '. If this integral ends up being more difficult, then the wrong choice has been made.

Common forms often arise; these, and others, are summarised in the table below ($n \in \mathbb{N}$ and $b, r \in \mathbb{R}$, $a \in \mathbb{R} \setminus \{0\}$).

| Integrand | $u$ (Differentiate) | $D\,v$ (Integrate) |
|---|---|---|
| $x^n\ e^{(ax\,+\,b)}$ | $x^n$ | $e^{(ax\,+\,b)}$ |
| $x^n\ \sin\,(ax + b)$ | $x^n$ | $\sin\,(ax + b)$ |
| $x^n\ \cos\,(ax + b)$ | $x^n$ | $\cos\,(ax + b)$ |
| $x^r\ \ln ax$ | $\ln ax$ | $x^r$ |
| $x\,\sqrt{ax + b}$ | $x$ | $\sqrt{ax + b}$ |

The first 3 in the above table are fairly common. Normally, $n = 1$. The case $n = 2$ sometimes arises and will be dealt with in the next subsection.

<u>Example 12</u>

Integrate $x \sin 4x$.

Differentiate the $x$ and integrate the $\sin 4x$. Set out the working as follows, remembering that $u$ and $v$ are functions of $x$ and that $D\,u$ means $u\,'(x)$ and similarly for $v$.

$$u\,(x) = x \quad , \quad v\,(x) =$$

$$u\,'(x) = \quad , \quad v\,'(x) = \sin 4x$$

Now differentiate $u$ and integrate $v\,'$ to get,

$$u\,(x) = x \quad , \quad v\,(x) = -\frac{1}{4}\cos 4x$$

$$\frac{d}{dx}\Bigg\downarrow \qquad\qquad\qquad \Bigg\uparrow\int$$

$$u'(x) = 1 \quad , \quad v'(x) = \sin 4x$$

Only the bottom right function ($v'$) from the above four is not used in the Integration by Parts Formula,

$$\int x \sin 4x \quad dx = x\left(-\frac{1}{4}\cos 4x\right) + \frac{1}{4}\int \cos 4x \quad dx$$

$$= -\frac{x}{4}\cos 4x + \frac{1}{4}\left(\frac{1}{4}\sin 4x\right) + C$$

$$= -\frac{x}{4}\cos 4x + \frac{1}{16}\sin 4x + C$$

It can be checked without too much effort that this alleged answer is correct by differentiating it and showing that the result is $x \sin 4x$.

Example 13

Integrate $x^{1/2} \ln 3x$.

The relevant bits and pieces are,

$$u(x) = \ln 3x \quad , \quad v(x) = \frac{2}{3}x^{3/2}$$

$$u'(x) = \frac{1}{x} \quad , \quad v'(x) = x^{1/2}$$

Putting these ingredients into the Integration by Parts recipe yields,

$$\int x^{1/2} \ln 3x \quad dx = \frac{2}{3}x^{3/2} \ln 3x - \frac{2}{3}\int x^{-1} x^{3/2} \quad dx$$

$$= \frac{2}{3}x^{3/2} \ln 3x - \frac{2}{3}\int x^{1/2} \quad dx$$

$$= \frac{2}{3}x^{3/2} \ln 3x - \frac{2}{3}\left(\frac{2}{3}x^{3/2}\right) + C$$

$$= \frac{2}{3}x^{3/2}\ \ln 3x \ - \ \frac{4}{9}x^{3/2} \ + \ C$$

## Repeated Integration by Parts

Sometimes the Integration by Parts procedure has to be repeated to get the answer. The process may have to be repeated more than once. However, normally in this course, only one repetition has to be performed.

Example 14

Integrate $x^2 \cos 3x$.

$$u(x) = x^2 \quad , \quad v(x) = \frac{1}{3}\sin 3x$$

$$u'(x) = 2x \quad , \quad v'(x) = \cos 3x$$

$$\int x^2 \cos 3x \ \ dx = x^2\left(\frac{1}{3}\sin 3x\right) - \frac{2}{3}\int x \sin 3x \ \ dx$$

The integral on the RHS needs to be integrated by parts. Be careful with letters here (maybe choose uppercase),

$$U(x) = x \quad , \quad V(x) = -\frac{1}{3}\cos 3x$$

$$U'(x) = 1 \quad , \quad V'(x) = \sin 3x$$

Check (very similar to Example 11) that,

$$\int x \sin 3x \ \ dx = -\frac{x}{3}\cos 3x + \frac{1}{9}\sin 3x$$

The constant of integration is not important here; just put it in after the very last integration step in the whole calculation.

Putting this back into the original equation gives,

$$\int x^2 \cos 3x \ dx = \frac{x^2}{3}\sin 3x - \frac{2}{3}\left(-\frac{x}{3}\cos 3x + \frac{1}{9}\sin 3x\right) + C$$

$$\int x^2 \cos 3x \ dx = \frac{x^2}{3}\sin 3x + \frac{2x}{9}\cos 3x - \frac{2}{27}\sin 3x + C$$

## The Hidden Function

When there only appears to be one function, there are, in fact, always two. The secret function is the constant function 1, which is always to be integrated.

Example 15

Evaluate $\int \ln x \ dx$.

Write this as,

$$\int 1 \,.\, \ln x \ dx$$

$$u(x) = \ln x \quad , \quad v(x) = x$$

$$u'(x) = \frac{1}{x} \quad , \quad v'(x) = 1$$

$$\begin{aligned} \int \ln x \ dx &= x \ln x - \int x^{-1} x \ dx \\ &= x \ln x - \int 1 \ dx \\ &= x \ln x - x + C \end{aligned}$$

Example 16

Evaluate $\int_0^{1/8} \sin^{-1} 4x \ dx$ correct to 6 d. p.

$$\int_0^{1/8} \sin^{-1} 4x \; dx = \int_0^{1/8} 1 \, . \, \sin^{-1} 4x \; dx$$

$$u(x) = \sin^{-1} 4x \quad , \quad v(x) = x$$

$$u'(x) = \frac{4}{\sqrt{1 - 16x^2}} \quad , \quad v'(x) = 1$$

$$\int_0^{1/8} \sin^{-1} 4x \; dx = \Big[ x \sin^{-1} 4x \Big]_0^{1/8} - \int_0^{1/8} \frac{4x}{\sqrt{1 - 16x^2}} \; dx$$

The integral on the RHS can be handled by a substitution of the form $p = 1 - 16x^2$. Hence, $dp = -32 \, x \, dx$. Then (check !),

$$\int_0^{1/8} \sin^{-1} 4x \; dx = \Big[ x \sin^{-1} 4x \Big]_0^{1/8} - \frac{1}{8} \int_{3/4}^{1} p^{-1/2} \; dp$$

$$= \frac{1}{8} \sin^{-1} \left( \frac{1}{2} \right) - 0 - \frac{1}{8} \Big[ 2\sqrt{p} \; \Big]_{3/4}^{1}$$

$$= \frac{1}{8} \left( \frac{\pi}{6} \right) - \frac{1}{8} \left( 2 - \sqrt{3} \right)$$

$$= \frac{\pi}{48} + \frac{\sqrt{3}}{8} - \frac{1}{4}$$

$$= 0 \cdot 031\,956 \quad \text{(to 6 d. p.)}$$

## *Cyclic Integrals*

If, in the course of Integration by Parts, the same integral appears after integrating, an equation for the unknown integral is solved.

<u>Example 17</u>

Find $\displaystyle\int e^{2x} \sin 3x \; dx$.

Denote the integral by $I$.

Which function do we differentiate ? Let's try differentiating the exponential,

$$u(x) = e^{2x} \quad , \quad v(x) = -\frac{1}{3}\cos 3x$$

$$u'(x) = 2e^{2x} \quad , \quad v'(x) = \sin 3x$$

$$I = -\frac{1}{3}e^{2x}\cos 3x + \frac{2}{3}\int e^{2x}\cos 3x \; dx$$

The integral on the RHS does not look any easier than the original one; so, we use Integration by Parts on this integral. Since we chose to differentiate the exponential before, we do this again (see what happens if we choose to integrate the exponential at this stage).

$$U(x) = e^{2x} \quad , \quad V(x) = \frac{1}{3}\sin 3x$$

$$U'(x) = 2e^{2x} \quad , \quad V'(x) = \cos 3x$$

$$\int e^{2x}\cos 3x \; dx = \frac{1}{3}e^{2x}\sin 3x - \frac{2}{3}\int e^{2x}\sin 3x \; dx$$

$$= \frac{1}{3}e^{2x}\sin 3x - \frac{2}{3}I$$

Putting this back into the main equation gives (remembering the constant),

$$I = -\frac{1}{3}e^{2x}\cos 3x + \frac{2}{3}\left(\frac{1}{3}e^{2x}\sin 3x - \frac{2}{3}I\right) + C$$

This is an algebraic equation in which we have to make $I$ the subject,

$$I = -\frac{1}{3}e^{2x}\cos 3x + \frac{2}{9}e^{2x}\sin 3x - \frac{4}{9}I + C$$

$$\frac{13}{9}I = -\frac{1}{3}e^{2x}\cos 3x + \frac{2}{9}e^{2x}\sin 3x + C$$

$$I = -\frac{3}{13}\, e^{2x} \cos 3x + \frac{2}{13}\, e^{2x} \sin 3x + C$$

## Reduction Formulae

Definition:

A **reduction formula** is a formula involving integrals where the integrand contains an expression raised to a natural number power.

The original integral in a reduction formula is normally written $I_n$. In this course, $I_n$ will usually be expressed in terms of either $I_{n-1}$ or $I_{n-2}$ (not both). A reduction formula is somewhat akin to a recurrence relation. Reduction formulae are used to work out the original integral for various values of *n*.

Example 18

Obtain a reduction formula for $I_n = \int x^n\, e^{2x}\, dx$ and hence find $I_0$, $I_1$ and $I_2$.

$$u(x) = x^n \quad , \quad v(x) = \frac{1}{2}\, e^{2x}$$

$$u'(x) = n\, x^{n-1} \quad , \quad v'(x) = e^{2x}$$

$$I_n = \frac{1}{2}\, x^n\, e^{2x} - \frac{n}{2} \int x^{n-1}\, e^{2x}\, dx$$

$$I_n = \frac{1}{2}\, x^n\, e^{2x} - \frac{n}{2}\, I_{n-1}$$

This is the desired reduction formula. Clearly,

$$I_0 = \frac{1}{2}\, e^{2x} + C$$

Using the reduction formula then gives,

$$I_1 = \frac{1}{2}\,x\,e^{2x} - \frac{1}{2}\,I_0$$

$$I_1 = \frac{1}{2}x\,e^{2x} - \frac{1}{2}\left(\frac{1}{2}\,e^{2x} + C\right)$$

$$I_1 = \frac{1}{2}x\,e^{2x} - \frac{1}{4}\,e^{2x} + D$$

where the constant has been relabelled ($D \equiv -C/2$). Also,

$$I_2 = \frac{1}{2}\,x^2\,e^{2x} - I_1$$

$$I_2 = \frac{1}{2}\,x^2\,e^{2x} - \left(\frac{1}{2}\,x\,e^{2x} - \frac{1}{4}\,e^{2x} + D\right)$$

$$I_2 = \frac{1}{2}\,x^2\,e^{2x} - \frac{1}{2}x\,e^{2x} + \frac{1}{4}\,e^{2x} + E$$

where again the constant has been relabeled ($E \equiv -D$).

## Differential Equations

Differential equations were introduced in Higher. The types studied there are very special cases of a more general class of differential equations.

### *Separable Differential Equations*

Definition:

A **separable differential equation** is one that can be written in the form,

$$\frac{dy}{dx} = f(x)\,g(y)$$

To solve a separable differential equation, any expression involving $x$ must be written on one side of the equation and any expression involving $y$ on the other. The differentials $dx$ and $dy$ play an important part here.

First, the equation is separated out as,

$$\frac{1}{g\,(y)}\ dy \ = \ f\,(x)\ dx$$

Then each side is integrated with respect to the relevant variable,

$$\int \frac{1}{g\,(y)}\ dy \ = \ \int f\ (x)\ \ dx$$

Once each side has been integrated, the equation is rearranged for $y$ in terms of $x$.

In Higher, the case where $g\,(y) \equiv 1$ was studied.

Example 19

Find the general solution of $\dfrac{dy}{dx} = 3y$.

$$\frac{dy}{dx} = 3y$$

$$\int \frac{1}{y}\ dy = \int 3\ \ dx$$

$$\ln |y| = 3x + C$$

$$|y| = e^{3x + C}$$

We have 2 situations to consider. If $y > 0$, then,

$$y = A\, e^{3x}$$

where the identification $A \equiv e^{C}$ has been made. Note that $A > 0$. If $y < 0$, then,

$$y = -A\, e^{3x}$$

There are thus 2 families of solutions. In most applications, the first one arises most often.

<u>Example 20</u>

Obtain the particular solution of the differential equation
$\frac{dP}{dy} = -\sin y \cos^2 3P$ given that $P = \frac{3\pi}{4}$ when $y = \frac{\pi}{2}$.

$$\frac{dP}{dy} = -\sin y \cos^2 3P$$

$$\int \frac{1}{\cos^2 3P}\, dP = -\int \sin y \, dy$$

Remembering the definition of secant,

$$\int \sec^2 3P \, dP = -\int \sin y \, dy$$

$$\frac{1}{3}\tan 3P = \cos y + C$$

Using the initial conditions to work out the constant of integration gives,

$$\frac{1}{3}\tan\frac{9\pi}{4} = \cos\frac{\pi}{2} + C$$

$$\frac{1}{3}.1 = 0 + C$$

$$C = \frac{1}{3}$$

Putting this back and solving for $P$ (not $y$!) gives,

$$\tan 3P = 3\cos y + 1$$

$$P = \frac{1}{3}\tan^{-1}(1 + 3\cos y)$$