Further Sequences and Series

Prerequisites: Differentiating functions; expanding brackets with many terms; adding, subtracting and simplifying fractions.

Maths Applications: Approximating differentiable functions by polynomials; calculating values of functions.

Real-World Applications: Operation of calculators; energy levels of atoms.

Power Series and D'Alembert's Ratio Test

Most of this section is for interest only, but the last definition and part of the last theorem have implications for the next section.

Theorem (**D'Alembert's Ratio Test** aka **Ratio Test**):

For a sequence $\{u_n\}$ of positive terms, the following hold,

$$\lim_{n\to\infty}\left(\frac{u_{n+1}}{u_n}\right) < 1 \Rightarrow \sum_{n=1}^{\infty} u_n \text{ converges}$$

$$\lim_{n\to\infty}\left(\frac{u_{n+1}}{u_n}\right) > 1 \Rightarrow \sum_{n=1}^{\infty} u_n \text{ diverges}$$

$$\lim_{n\to\infty}\left(\frac{u_{n+1}}{u_n}\right) = 1 \Rightarrow \text{no conclusion}$$

Example 1

Show that the series $\sum_{n=1}^{\infty} \frac{n}{7^n}$ converges.

We have $u_n = \frac{n}{7^n}$ and $u_{n+1} = \frac{n+1}{7^{n+1}}$. Hence,

$$\lim_{n\to\infty}\left(\frac{u_{n+1}}{u_n}\right) = \lim_{n\to\infty}\left(\frac{7^n\,(n+1)}{7^{n+1}\,n}\right)$$

$$= \lim_{n\to\infty}\left(\frac{(n+1)}{7n}\right)$$

$$= \lim_{n\to\infty}\left(\frac{(1 + 1/n)}{7}\right)$$

$$= \frac{1}{7}$$

Hence, as $\lim_{n\to\infty}\left(\frac{u_{n+1}}{u_n}\right) = \frac{1}{7} < 1$, the Ratio Test implies that the series $\sum_{n=1}^{\infty}\frac{n}{7^n}$ converges.

<u>Example 2</u>

Show that $\sum_{n=1}^{\infty}\frac{5^n}{2^n + 1}$ diverges.

The relevant terms for the limit are $u_n = \frac{5^n}{2^n + 1}$ and $u_{n+1} = \frac{5^{n+1}}{2^{n+1} + 1}$. Hence,

$$\lim_{n\to\infty}\left(\frac{u_{n+1}}{u_n}\right) = \lim_{n\to\infty}\left(\frac{5^{n+1}}{2^{n+1} + 1} \times \frac{2^n + 1}{5^n}\right)$$

$$= \lim_{n\to\infty}\left(\frac{5\,(2^n + 1)}{(2^{n+1} + 1)}\right)$$

Dividing numerator and denominator by 2^n gives,

$$\lim_{n\to\infty}\left(\frac{u_{n+1}}{u_n}\right) = \frac{5}{2}$$

By the Ratio Test, $\sum_{n=1}^{\infty} \frac{5^n}{2^n + 1}$ diverges.

For the next example, the result that $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges to $\frac{\pi^2}{6}$ will be needed.

Example 3

Show that $\lim_{n\to\infty} \left(\frac{u_{n+1}}{u_n}\right) = 1$ for the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ and that the series converges.

We know that the series converges to $\frac{\pi^2}{6}$. It remains to show that $\lim_{n\to\infty} \left(\frac{u_{n+1}}{u_n}\right) = 1$. We have $u_n = \frac{1}{n^2}$ and $u_{n+1} = \frac{1}{(n+1)^2}$. Hence,

$$\lim_{n\to\infty} \left(\frac{u_{n+1}}{u_n}\right) = \lim_{n\to\infty} \left(\frac{n^2}{(n+1)^2}\right)$$

$$= \lim_{n\to\infty} \left(\frac{n^2}{n^2 + 2n + 1}\right)$$

$$= \lim_{n\to\infty} \left(\frac{1}{1 + {}^2/_n + {}^1/_{n^2}}\right)$$

$$= 1$$

Example 4

Show that $\lim_{n\to\infty} \left(\frac{u_{n+1}}{u_n}\right) = 1$ for the series $\sum_{n=1}^{\infty} 1$ and that the series diverges.

Clearly, $u_n = u_{n+1} = 1$. Hence, $\lim_{n \to \infty} \left(\frac{u_{n+1}}{u_n} \right) = 1$. The series clearly does not converge; hence, it diverges.

Quite often, a series will have some negative terms. These need to be analysed.

<u>Definition:</u>

A series $\sum_{n=1}^{\infty} u_n$ is **absolutely convergent** if $\sum_{n=1}^{\infty} |u_n|$ converges.

<u>Example 5</u>

Show that $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n!}$ is absolutely convergent.

We will use the Ratio Test on the series $\sum_{n=1}^{\infty} \left| \frac{(-1)^{n+1}}{n!} \right| = \sum_{n=1}^{\infty} \frac{1}{n!}$.

So, $|u_n| = \frac{1}{n!}$ and $|u_{n+1}| = \frac{1}{(n+1)!}$. The limit is,

$$\lim_{n \to \infty} \left(\frac{|u_{n+1}|}{|u_n|} \right) = \lim_{n \to \infty} \left(\frac{1}{(n+1)!} \times \frac{n!}{1} \right)$$

$$= \lim_{n \to \infty} \left(\frac{1}{n+1} \right)$$

$$= 0$$

Hence, as the limit is less than 1, $\sum_{n=1}^{\infty} \left| \frac{(-1)^{n+1}}{n!} \right|$ is convergent. Thus, $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n!}$ is absolutely convergent.

Theorem:

Any absolutely convergent series is convergent.

This theorem just says that if the modulus of each term of a series is taken and all the terms added together make this new series converge, then surely the original series (with some terms possibly negative, thus making the sum smaller) must also converge.

Example 6

Show that $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n!}$ is convergent.

By Example 5, the series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n!}$ is absolutely convergent. The previous theorem then shows that $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n!}$ is convergent.

Definition:

A series $\sum_{n=1}^{\infty} u_n$ is **conditionally convergent** if $\sum_{n=1}^{\infty} u_n$ converges but $\sum_{n=1}^{\infty} |u_n|$ diverges.

Example 7

Given that the series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ converges to ln 2, show that $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ is conditionally convergent.

As the original series converges, it remains to show that $\sum_{n=1}^{\infty} \left| \frac{(-1)^{n+1}}{n} \right|$ diverges. This is a standard argument, but not necessarily obvious at first sight, and proceeds as follows. Writing out the last series mentioned,

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots$$

and comparing it with the series,

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{4} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \ldots$$

where the next 8 terms are $+\frac{1}{16}$, the 16 after that are $+\frac{1}{32}$, etc…, it is easily seen that the sum of the first series is bigger than the second and the second clearly diverges (adding infinitely many halves). Thus, the first series must diverge too.

Hence, as $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ converges but $\sum_{n=1}^{\infty} \left| \frac{(-1)^{n+1}}{n} \right|$ diverges, $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ is conditionally convergent.

There are some weird theorems regarding absolutely convergent and conditionally convergent series, which, although not required for this course, will be stated for interest.

<u>Theorem:</u>

Any rearrangement of an absolutely convergent series converges to the same limit.

<u>Theorem (**Riemann's Rearrangement Theorem**):</u>

Any conditionally convergent series can be rearranged to converge to any real number, or rearranged to diverge.

There is a d'Alembert's Ratio Test for series with negative terms.

<u>Theorem (**D'Alembert's Ratio Test for Absolute Convergence**):</u>

For a sequence $\{u_n\}$(of not necessarily all positive terms), the following hold,

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| < 1 \;\Rightarrow\; \sum_{n=1}^{\infty} u_n \text{ converges absolutely}$$

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| > 1 \;\Rightarrow\; \sum_{n=1}^{\infty} u_n \text{ diverges}$$

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = 1 \;\Rightarrow\; \text{no conclusion}$$

<u>Example 8</u>

Show that $\displaystyle\sum_{n=1}^{\infty} \frac{n\,(-1)^n}{e^n}$ converges absolutely.

The limit can be written as,

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left(1 + \frac{1}{n}\right)\frac{1}{e} = \frac{1}{e}$$

Hence, the given series is absolutely convergent.

<u>Example 9</u>

Show that $\displaystyle\sum_{n=1}^{\infty} \frac{(-10)^n}{5^{2n+1}\,(n+3)}$ converges.

It suffices to show that the given series is absolutely convergent. The limit can be written as,

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\frac{2}{5}\left(\frac{1 + 3/n}{1 + 4/n}\right) = \frac{2}{5}$$

Thus, the given series is absolutely convergent and thus convergent.

Example 10

Show that $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^n e^n}{n}$ diverges.

From Example 8, it is easily seen that,

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = e$$

and hence that the given series diverges.

Example 11

Show that $\displaystyle\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = 1$ for the series $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}$ and that the series converges absolutely.

By a similar argument to that given in Example 3, it is found that

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left(\frac{1}{1 + 2/n + 1/n^2}\right) = 1$$

By the comment made before Example 3, $\displaystyle\sum_{n=1}^{\infty}\left|\frac{(-1)^n}{n^2}\right| = \sum_{n=1}^{\infty}\frac{1}{n^2}$, which converges to $\dfrac{\pi^2}{6}$. Hence, $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}$ is absolutely convergent.

Example 12

Show that $\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = 1$ for the series $\sum_{n=1}^{\infty}(-1)^n$ and that the series diverges.

The limit is,

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{(-1)^{n+1}}{(-1)^n}\right|$$

$$= \lim_{n\to\infty}\left|(-1)\right|$$

$$= 1$$

The partial sums jump from -1 to 0 to -1 to 0 etc… . Hence, the series does not converge and so diverges.

Example 13

Show that $\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = 1$ for the series $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}$ and that the series converges conditionally.

From Example 7, the series $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}$ is conditionally convergent.

The limit is,

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left(\frac{1}{1+\frac{1}{n}}\right) = 1$$

Definition:

The **interval of convergence** (aka **radius of convergence**) of the power series,

$$\sum_{i=0}^{\infty} a_i x^i = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots \quad (a_i \in \mathbb{R})$$

is the set of x values for which the series converges.

A special case of d'Alembert's Ratio Test for power series is worth stating as a theorem.

Theorem (**D'Alembert's Ratio Test for Power Series**):

For the power series $\sum_{n=0}^{\infty} a_n x^n$,

$$\lim_{n \to \infty} \left| \frac{a_{n+1}\, x}{a_n} \right| < 1 \Rightarrow \sum_{n=0}^{\infty} a_n x^n \text{ converges absolutely}$$

$$\lim_{n \to \infty} \left| \frac{a_{n+1}\, x}{a_n} \right| > 1 \Rightarrow \sum_{n=0}^{\infty} a_n x^n \text{ diverges}$$

$$\lim_{n \to \infty} \left| \frac{a_{n+1}\, x}{a_n} \right| = 1 \Rightarrow \text{no conclusion}$$

Example 14

Find the sum to infinity of the series $1 + 3x + 5x^2 + 7x^3 + \ldots$, stating the values of x for which this infinite sum exists.

We have,

$$S_n = 1 + 3x + 5x^2 + 7x^3 + \ldots + (2n - 1)x^{n-1}$$

Multiplying by x gives,

$$xS_n = x + 3x^2 + 5x^3 + 7x^4 + \ldots + (2n - 1)x^n$$

Subtracting,

$$(1 - x)S_n = 1 + 2x + 2x^2 + 2x^3 + \ldots + 2x^{n-1} - (2n - 1)x^n$$

$$(1 - x)S_n = 1 + 2(x + x^2 + x^3 + \ldots + x^{n-1}) - (2n - 1)x^n$$

Assuming that $-1 < x < 1$,

$$(1 - x)S_n = 1 + 2\left(\frac{x(1 - x^{n-1})}{1 - x}\right) - (2n - 1)x^n$$

$$S_n = \frac{1}{1 - x} + 2\left(\frac{x(1 - x^{n-1})}{(1 - x)^2}\right) - \frac{(2n - 1)x^n}{1 - x}$$

Given that $-1 < x < 1$, as $n \to \infty$,

$$S_\infty = \frac{1}{1 - x} + \frac{2x}{(1 - x)^2}$$

$$S_\infty = \frac{1 + x}{(1 - x)^2} \qquad (-1 < x < 1)$$

<u>Example 15</u>

State the values of x for which the series $1 + 3x + 5x^2 + 7x^3 + \ldots$ diverges.

From Example 14, the series converges for $(-1 < x < 1)$. Hence, the series diverges for $x \leq -1$ and $x \geq 1$.

<u>Example 16</u>

Show that $\lim_{n \to \infty}\left|\frac{a_{n+1}\,x}{a_n}\right| = 1$ for $x = 1$ for the series $\sum_{n=1}^{\infty} \frac{x^n}{n^2}$ and the series converges.

Using the details in Example 3, for $x = 1$,

$$\lim_{n\to\infty} \left| \frac{a_{n+1}x}{a_n} \right| = \lim_{n\to\infty} \left(\frac{1}{1 + \frac{2}{n} + \frac{1}{n^2}} \right) = 1$$

From Example 3, the series is convergent.

Example 17

Show that $\lim_{n\to\infty} \left| \frac{a_{n+1}\ x}{a_n} \right| = 1$ for $x = 1$ for the series $\sum_{n=1}^{\infty} x^n$ and the series diverges.

The limit clearly equals 1 and the series is $1 + 1 + 1 + 1 + \ldots$, which clearly diverges.

The important part of d'Alembert's Ratio Test is the case of absolute convergence, which implies convergence.

Maclaurin Series

Definition:

The **Maclaurin series** (aka **Maclaurin expansion**) of a function f is the power series,

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$$

In expanded form, this is,

$$f(x) = f(0) + f'(0)\,x + f''(0)\,\frac{x^2}{2!} + f'''(0)\,\frac{x^3}{3!} + \ldots$$

Standard Series

The following Maclaurin series must be known (try deriving them). D'Alembert's Ratio Test for Power Series is used to obtain the values of x for which the expansion is valid (the radius of convergence).

<u>Theorem:</u>

The Maclaurin series for e^x is,

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

and is valid $\forall x \in \mathbb{R}$.

<u>Theorem:</u>

The Maclaurin series for $\sin x$ is,

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}$$

and is valid $\forall x \in \mathbb{R}$.

<u>Theorem:</u>

The Maclaurin series for $\cos x$ is,

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}$$

and is valid $\forall x \in \mathbb{R}$.

<u>Theorem:</u>

The Maclaurin series for $\tan^{-1} x$ is,

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1}$$

and is valid for $-1 \leq x \leq 1$.

<u>Theorem:</u>

The Maclaurin series for $\ln(1 + x)$ is,

$$\ln(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} x^n$$

and is valid for $-1 < x \leq 1$.

The reason for considering $\ln(1 + x)$ instead of $\ln x$ is that derivatives of $\ln x$ will have powers of x in the denominator and then x has to be put equal to zero (thus a zero denominator will exist).

Modified Standard Series

Once the standard Maclaurin series are known, slight modifications of them can be obtained by replacing x with $2x$, $3x$, etc.

<u>Example 18</u>

Find the first 5 terms of the Maclaurin series for e^{2x}.

The series for e^{2x} is the series for e^x with x replaced with $2x$,

$$e^{2x} = 1 + (2x) + \frac{(2x)^2}{2!} + \frac{(2x)^3}{3!} + \frac{(2x)^4}{4!} + \dots$$

The brackets must be expanded and the numbers simplified,

$$e^{2x} = 1 + 2x + \frac{4\,x^2}{2} + \frac{8\,x^3}{6} + \frac{16\,x^4}{24} + \ldots$$

$$e^{2x} = 1 + 2\,x + 2x^2 + \frac{4\,x^3}{3} + \frac{2\,x^4}{3} + \ldots$$

That's it. Don't decimalise the fractions.

Example 19

Find the first 4 non-zero terms of the Maclaurin series for $\sin 3x$.

$$\sin 3x = (3x) - \frac{(3x)^3}{3!} + \frac{(3x)^5}{5!} - \frac{(3x)^7}{7!} + \ldots$$

$$\sin 3x = 3x - \frac{27\,x^3}{6} + \frac{243\,x^5}{120} - \frac{2\,187\,x^7}{5\,040} + \ldots$$

$$\sin 3x = 3\,x - \frac{9\,x^3}{2} + \frac{81\,x^5}{40} - \frac{243\,x^7}{560} + \ldots$$

Example 20

Find the first 6 terms of the Maclaurin series for $\cos 4x$.

$$\cos 4x = 1 - \frac{(4x)^2}{2!} + \frac{(4x)^4}{4!} - \ldots$$

The 2nd term involving x has coefficient 0, as do the 4th (term in x^3) and 6th (term in x^5), so we need not go any further,

$$\cos 4x = 1 - 8\,x^2 + \frac{32\,x^4}{3} - \ldots$$

Example 21

Find the first 4 non-zero terms of the Maclaurin series for $\tan^{-1} 6x$.

$$\tan^{-1} 6x = (6x) - \frac{(6x)^3}{3} + \frac{(6x)^5}{5} - \frac{(6x)^7}{7} + \ldots$$

$$\tan^{-1} 6x = 6x - 72x^3 + \frac{7\,776\,x^5}{5} - \frac{279\,936\,x^7}{7} + \ldots$$

<u>Example 22</u>

Find the first 4 terms of the Maclaurin series for $\ln(1 + 5x)$.

$$\ln(1 + 5x) = (5x) - \frac{(5x)^2}{2} + \frac{(5x)^3}{3} - \frac{(5x)^4}{4} + \ldots$$

$$\ln(1 + 5x) = 5x - \frac{25x^2}{2} + \frac{125x^3}{3} - \frac{625x^4}{4} + \ldots$$

<u>Example 23</u>

Express $e^{x/4}$ in the form $\displaystyle\sum_{n=0}^{\infty} \frac{1}{n!} k^n x^n$, stating the value of the integer k.

$$e^{x/4} = \sum_{n=0}^{\infty} \frac{(x/4)^n}{n!}$$

$$= \sum_{n=0}^{\infty} \frac{x^n}{4^n}\frac{1}{n!}$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!}\frac{1}{4^n} x^n$$

Hence, $k = \frac{1}{4}$.

Combinations of Standard Series

<u>Example 24</u>

Find the Maclaurin series for $e^{2x} \sin 3x$ up to the term involving x^4.

$$e^{2x}\sin 3x = \left(1 + 2x + 2x^2 + \frac{4x^3}{3} + \ldots\right) \times \left(3x - \frac{9x^3}{2} + \ldots\right)$$

There is no need to consider higher terms in e^{2x}, as any other higher terms multiplied by the second bracket will give terms involving at least x^6. Similarly, there is no need to consider higher terms in sin $3x$, as any other higher terms multiplied by the first bracket will give terms involving at least x^5. So, expanding the brackets gives,

$$e^{2x}\sin 3x = 3x + 6x^2 + 6x^3 + 4x^4 + \ldots - \frac{9x^3}{2} - 18x^4 - \ldots$$

$$e^{2x}\sin 3x = 3x + 6x^2 + \frac{3x^3}{2} - 5x^4 + \ldots$$

<u>Example 25</u>

Find the first 6 terms in the Maclaurin expansion for $\cos 2x \,.\, \tan^{-1} x$.

Let $f(x) = \cos 2x \,.\, \tan^{-1} x$. Then,

$$f(x) = \left(1 - 2x^2 + \frac{2x^4}{3} - \ldots\right) \times \left(x - \frac{x^3}{3} + \frac{x^5}{5} - \ldots\right)$$

$$= x - 2x^3 + \frac{2x^5}{3} - \ldots - \frac{x^3}{3} + \frac{2x^5}{3} - \ldots + \frac{x^5}{5} - \ldots$$

$$= x - \frac{7x^3}{3} + \frac{23x^5}{15} - \ldots$$

<u>Example 26</u>

Find the Maclaurin series for e^{3x+x^2} up to the term in x^5.

The starting point here is to note that $e^{3x+x^2} = e^{3x}e^{x^2}$. So, write down the Maclaurin series for e^{3x} and e^{x^2},

$$e^{3x} = 1 + 3x + \frac{9x^2}{2} + \frac{27x^3}{6} + \frac{81x^4}{24} + \frac{243x^5}{120} + \ldots$$

$$e^{x^2} = 1 + x^2 + \frac{x^4}{2} \quad \ldots$$

Multiplying these gives (note the setting out),

$$e^{3x+x^2} = 1 + 3x + \frac{9x^2}{2} + \frac{27x^3}{6} + \frac{81x^4}{24} + \frac{243x^5}{120} + \ldots$$

$$+ x^2 + 3x^3 + \frac{9x^4}{2} + \frac{27x^5}{6} + \ldots$$

$$+ \frac{x^4}{2} + \frac{3x^5}{2} + \ldots$$

Collecting the terms together gives (check),

$$e^{3x+x^2} = 1 + 3x + \frac{11x^2}{2} + \frac{15x^3}{2} + \frac{67x^4}{8} + \frac{321x^5}{40} + \ldots$$

<u>Example 27</u>

Find the first 5 terms in the Maclaurin series for $\ln(1 - 9x^2)$.

This can be done in 2 ways. The first method involves replacing x with $-9x^2$ in the Maclaurin series for $\ln(1 + x)$. Try this. The second method involves rewriting $\ln(1 - 9x^2)$ as,

$$\ln(1 - 9x^2) = \ln[(1 - 3x)(1 + 3x)]$$

$$= \ln(1 - 3x) + \ln(1 + 3x)$$

and then using a Maclaurin series for both logarithmic terms. We have,

$$\ln(1 - 3x) = -3x - \frac{9x^2}{2} - \frac{27x^3}{3} - \frac{81x^4}{4} - \ldots$$

$$\ln(1 + 3x) = 3x - \frac{9x^2}{2} + \frac{27x^3}{3} - \frac{81x^4}{4} - \ldots$$

Adding these gives,

$$\ln(1 - 3x) + \ln(1 + 3x) = -9x^2 - \frac{81x^4}{2} - \ldots$$

<u>Example 28</u>

Find the Maclaurin series for ln (cos x) up to the term in x^6.

The trick here is to write cos x as 1 plus something. This is easy, as cos x = 1 + (cos x − 1). Then, replacing x with (cos x − 1) in the Maclaurin expansion for ln (1 + x) and remembering that we only need powers of x up to 6, we get

$$\begin{aligned}\ln(1 + (\cos x - 1)) &= \left(-\frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots\right) \\ &\quad - \frac{1}{2}\left(-\frac{x^2}{2!} + \frac{x^4}{4!} - \ldots\right)^2 \\ &\quad + \frac{1}{3}\left(-\frac{x^2}{2!} + \ldots\right)^3 - \ldots\end{aligned}$$

Upon simplification, this becomes, ignoring terms bigger than x^6,

$$\ln(\cos x) = -\frac{x^2}{2} - \frac{x^4}{12} - \frac{x^6}{45} - \ldots$$

Other Maclaurin Series from Scratch

Some Maclaurin series have to be obtained from scratch. This involves differentiating the given function repeatedly and putting $x = 0$.

<u>Example 29</u>

Find the Maclaurin expansion for $\sqrt{1 + x}$ up to the term involving x^4.

Let $f(x) = \sqrt{1 + x}$. Then (check),

$$f(x) = \sqrt{1 + x} \quad \Rightarrow \quad f(0) = 1$$

$$f'(x) = \frac{1}{2\sqrt{1 + x}} \quad \Rightarrow \quad f'(0) = \frac{1}{2}$$

$$f''(x) = -\frac{1}{4}\left(\frac{1}{\sqrt{1+x}}\right)^3 \quad \Rightarrow \quad f''(0) = -\frac{1}{4}$$

$$f'''(x) = \frac{3}{8}\left(\frac{1}{\sqrt{1+x}}\right)^5 \quad \Rightarrow \quad f'''(0) = \frac{3}{8}$$

$$f''''(x) = -\frac{15}{16}\left(\frac{1}{\sqrt{1+x}}\right)^7 \quad \Rightarrow \quad f''''(0) = -\frac{15}{16}$$

Substituting these values into,

$$f(x) = f(0) + f'(0)\,x + f''(0)\,\frac{x^2}{2!} + f'''(0)\,\frac{x^3}{3!} + \ldots$$

gives,

$$\sqrt{1+x} = 1 + \frac{1}{2}\,.\,x - \frac{1}{4}\,.\,\frac{x^2}{2!} + \frac{3}{8}\,.\,\frac{x^3}{3!} - \frac{15}{16}\,.\,\frac{x^4}{4!} + \ldots$$

Simplifying this mess gives,

$$\sqrt{1+x} = 1 + \frac{x}{2} - \frac{x^2}{8} + \frac{x^3}{16} - \frac{5x^4}{128} + \ldots$$

<u>Example 30</u>

Find the first 2 non-zero terms in the Maclaurin series for tan x.

Let $f(x) = \tan x$. Then (check),

$$f(x) = \tan x \quad \Rightarrow \quad f(0) = 0$$

$$f'(x) = \sec^2 x \quad \Rightarrow \quad f'(0) = 1$$

$$f''(x) = 2\sec^2 x \tan x \quad \Rightarrow \quad f''(0) = 0$$

$$f'''(x) = 4\sec^2 x \tan^2 x + 2\sec^4 x \quad \Rightarrow \quad f'''(0) = 2$$

Then,

$$\tan x = 0 + 1 \,.\, x + 0 \,.\, \frac{x^2}{2!} + 2 \,.\, \frac{x^3}{3!} + \ldots$$

$$\tan x = x + \frac{x^3}{3} + \ldots$$

Example 31

Find the first 5 terms in the Maclaurin expansion for $\dfrac{e^x}{\cos x}$.

Don't even think about dividing 2 infinite series. We want to write $\dfrac{e^x}{\cos x}$ in the form,

$$\frac{e^x}{\cos x} = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4 + \ldots$$

where all the c_i are real constants.

Multiplying both sides of the above equation by $\cos x$ gives,

$$e^x = \cos x \,(c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4 + \ldots)$$

Substituting the Maclaurin expansion for $\cos x$ (up to x^4) into the above equation and expanding the brackets allows the RHS to be written as,

$$e^x = d_0 + d_1 x + d_2 x^2 + d_3 x^3 + d_4 x^4 + \ldots$$

where,

$$d_0 = c_0$$

$$d_1 = c_1$$

$$d_2 = c_2 - \frac{c_0}{2!}$$

$$d_3 = c_3 - \frac{c_1}{2!}$$

$$d_4 = c_4 - \frac{c_2}{2!} + \frac{c_0}{4!}$$

Comparing the above expression for e^x with the known Maclaurin series for e^x shows that,

$$d_0 = 1$$

$$d_1 = 1$$

$$d_2 = \frac{1}{2}$$

$$d_3 = \frac{1}{6}$$

$$d_4 = \frac{1}{24}$$

Comparing the c_i with the d_i allows us to solve successively for the c_i as,

$$c_0 = 1$$

$$c_1 = 1$$

$$c_2 = 1$$

$$c_3 = \frac{2}{3}$$

$$c_4 = \frac{1}{2}$$

Finally, we have,

$$\frac{e^x}{\cos x} = 1 + x + x^2 + \frac{2}{3} x^3 + \frac{1}{2} x^4 + \dots$$

The Binomial Series

<u>Definition:</u>

The **binomial series** is the Maclaurin expansion of $(1 + x)^r$.

<u>Theorem:</u>

The binomial series is,

$$(1 + x)^r = \sum_{k=0}^{\infty} \binom{r}{k} x^k$$

and is valid $\forall\, r \in \mathbb{R}$.

The binomial series finds application in physics.

<u>Example 32</u>

The kinetic energy of a relativistic particle is given by $E_K = m_0 c^2 \left(1 - \frac{v^2}{c^2}\right)^{-1/2} - m_0 c^2$.

Show that for $v \ll c$, so that terms involving v^4 and higher can be ignored, the kinetic energy is approximated by $\frac{1}{2} m_0 v^2$.

Expanding $\left(1 - \frac{v^2}{c^2}\right)^{-1/2}$ gives,

$$\left(1 - \frac{v^2}{c^2}\right)^{-1/2} \approx 1 + \left(-\frac{1}{2}\right)\left(-\frac{v^2}{c^2}\right) + \left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(\frac{1}{2!}\right)\left(-\frac{v^2}{c^2}\right)^2$$

$$= 1 + \frac{v^2}{2c^2} + \frac{3v^4}{8c^4}$$

Hence,

$$E_K \approx m_0c^2\left(1 + \frac{v^2}{2c^2} + \frac{3v^4}{8c^4}\right) - m_0c^2$$

$$= m_0c^2 + \frac{m_0v^2}{2} - m_0c^2$$

$$= \frac{1}{2}m_0v^2$$

Using Maclaurin Series to Estimate Values of Functions

Example 33

Find $\cos 0\cdot 5$ correct to 4 d.p. .

To find how many terms of the Maclaurin series we need to consider, it suffices to consider the first term that gives 0 to 5 d.p.. Simple investigation reveals that the first such term is $\frac{(0\cdot 5)^8}{8!}$. Hence,

$$\cos\ 0\cdot 5 \approx 1 - \frac{(0\cdot 5)^2}{2!} + \frac{(0\cdot 5)^4}{4!} - \frac{(0\cdot 5)^6}{6!} = 0\cdot 877\ 6$$

Using Maclaurin Series to Derive Values for Infinite Sums

As a special treat, in this final section on Maclaurin series, some funky expressions involving π will be derived.

Example 34

Putting $x = 1$ in the Maclaurin series for $\tan^{-1} x$,

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots$$

$$\tan^{-1} 1 = 1 - \frac{1^3}{3} + \frac{1^5}{5} - \frac{1^7}{7} + \ldots$$

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots$$

Example 35

Putting $x = \pi$ in the Maclaurin series for $\sin x$,

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \ldots$$

$$\sin \pi = \pi - \frac{\pi^3}{3!} + \frac{\pi^5}{5!} - \frac{\pi^7}{7!} + \frac{\pi^9}{9!} - \ldots$$

$$0 = \pi - \frac{\pi^3}{3!} + \frac{\pi^5}{5!} - \frac{\pi^7}{7!} + \frac{\pi^9}{9!} - \ldots$$

Differentiation and Integration of Maclaurin Series

Power series may be differentiated and integrated according to the following theorems.

Theorem (**Differentiation of Power Series**):

If $f(x) = \sum_{r=0}^{\infty} a_r x^r$ has radius of convergence $R > 0$, then, for $|x| < R$, f is differentiable and

$$f'(x) = \sum_{r=0}^{\infty} r\, a_r\, x^{r-1}$$

and has radius of convergence R.

<u>Theorem (**Integration of Power Series**):</u>

If $f(x) = \sum_{r=0}^{\infty} a_r x^r$ has radius of convergence $R > 0$, then, for $|x| < R$, f is integrable and

$$\int f(x)\,dx = \sum_{r=1}^{\infty} \frac{a_r\, x^{r+1}}{r+1}$$

and has radius of convergence R.

<u>Example 36</u>

Find the Maclaurin series for $\frac{1}{1+x^2}$ up to the term in x^6.

Since $\frac{1}{1+x^2} = \frac{d}{dx} \tan^{-1} x$, differentiating the Maclaurin series for $\tan^{-1} x$ gives,

$$\frac{1}{1+x^2} = \frac{d}{dx}\left(x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots \right)$$

$$\frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + \cdots$$

<u>Example 37</u>

Show using Maclaurin series that $\int \sin x\, dx = -\cos x + C$.

We have,

$$\int \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots \right) dx = \frac{x^2}{2} - \frac{x^4}{4.3!} + \frac{x^6}{6.5!} - \ldots + D$$

$$= -1 + \frac{x^2}{2} - \frac{x^4}{4!} + \frac{x^6}{6!} - \ldots + 1 + D$$

$$= -\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots\right) + C$$

$$= -\cos x + C$$

where the constant $D + 1$ has been relabelled as C.

Iteration and Fixed Points

Definition:

An **iterative scheme** is a recurrence relation.

Definition:

An **iterative sequence** is a sequence generated by an iterative scheme.

Definition:

A **fixed point** (aka **convergent** or **limit**) of an iterative scheme is a point a satisfying,

$$a = F(a)$$

Example 38

Find the fixed point(s) of the iterative scheme $x_{n+1} = \frac{1}{11}\left(x_n + \frac{2}{x_n}\right)$.

Any fixed points occur as x_n and x_{n+1} approach a fixed number L. So, as $n \to \infty$, $x_n \to L$ and $x_{n+1} \to L$. Hence,

$$L = \frac{1}{11}\left(L + \frac{2}{L}\right)$$

$$11L = L + \frac{2}{L}$$

$$10L = \frac{2}{L}$$

$$10L^2 = 2$$

$$L = \pm\frac{1}{\sqrt{5}}$$

Thus, the fixed points are $\pm\frac{1}{\sqrt{5}}$.