Integral Calculus

Prerequisites: Integrating $(ax+b)^n$, $\sin(ax+b)$ and $\cos(ax+b)$; derivatives of $\tan x$, $\sec x$, $\operatorname{cosec} x$, $\cot x$, e^x and $\ln x$; sum/difference rules; areas under and between curves.

Maths Applications: Calculating areas and volumes of standard geometric figures.

Real-World Applications: Particle motion.

Standard Integrals

From the new derivatives obtained in the previous topic, we obtain a handful of new indefinite integrals for free.

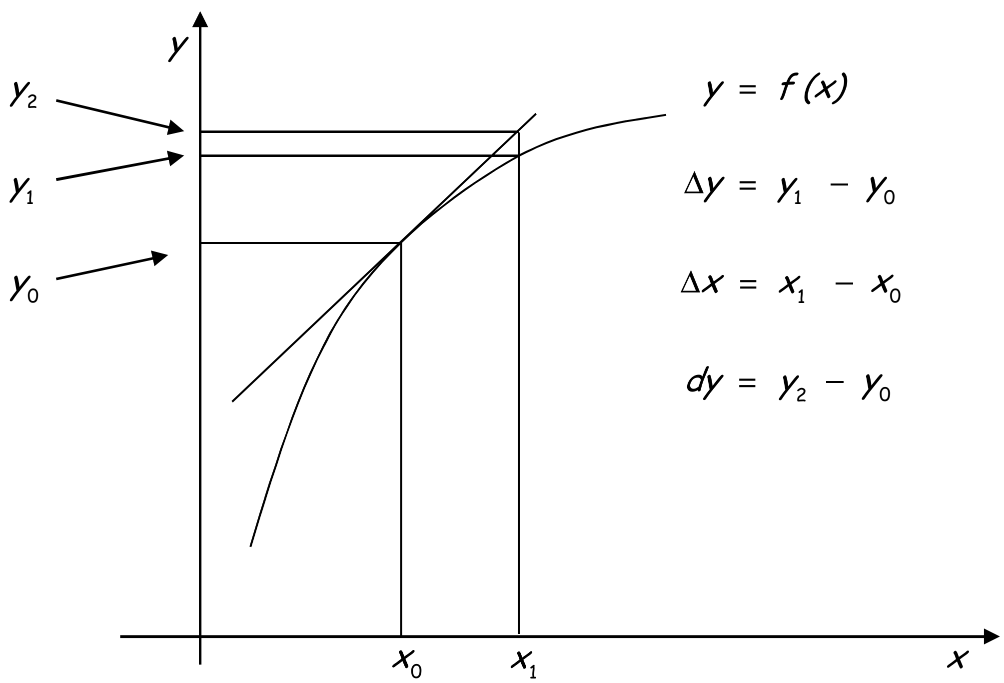
- $\int e^x \, dx = e^x + C$
- $\int \frac{1}{x} \, dx = \ln|x| + C$
- $\int \sec^2 x \, dx = \tan x + C$
- $\int \sec x \tan x \, dx = \sec x + C$
- $\int \operatorname{cosec} x \cot x \, dx = -\operatorname{cosec} x + C$
- $\int \operatorname{cosec}^2 x \, dx = -\cot x + C$

In the second integral, $|x| = x$ $(x \geq 0)$ and $|x| = -x$ $(x < 0)$. In other words, $|x|$ is always 0 or positive.

Differentials

For a function $y = f(x)$, $\frac{dy}{dx}$ is the rate of change of y with respect to x. The quantities dy and dx in this expression do not have meanings on their own. However, confusingly, we can introduce quantities like these which do have an independent meaning.

A small incremental change in x, denoted by Δx, will give rise to a small incremental change in y, denoted Δy. The derivative is the limit of $\Delta y / \Delta x$ as $\Delta x \to 0$. We are interested in a slightly different quantity, namely, the change in the y – values along the tangent line.



<u>Definition:</u>

For a function $y = f(x)$, the **differential** of f is the function defined by,

$$dy \overset{def}{=} f'(x)\,\Delta x$$

Considering the function $y = x$, the definition gives $dx = \Delta x$. This leads to an alternative way of writing the differential of f. Another way just involves using $\frac{dy}{dx}$.

<u>Corollary:</u>

The differential of f can be written in 2 other ways,

$$dy = f\,'(x)\ dx$$

$$dy = \frac{dy}{dx}\ dx$$

It is the second expression in the corollary that leads to a common misconception that the 'dx 's cancel out'; they don't cancel out, and technically $\frac{dy}{dx}$ *is not to be treated as a fraction* (because it ain't). However, in practice, we can get away with treating it as though it were a fraction.

<u>Example 1</u>

Find dy for the function $y = x^2$.

$$\frac{dy}{dx} = 2x$$

$$dy = 2x\,dx$$

<u>Example 2</u>

Find du for the function $u = x^2 + 3x$.

$$\frac{du}{dx} = 2x + 3$$

$$du = (2x + 3)\ dx$$

<u>Integration by Substitution</u>

It is not always obvious how to integrate a function. Sometimes, however, the function can be rewritten through a change of variable (i.e., a substitution) in a form that is a standard integral. The process involved in getting the standard integral is called **integration by substitution**.

Theorem (**Change of Variables Theorem**):

$$\int_a^b f\,((u\,(x))\,u\,'(x)\;dx \;=\; \int_c^d f\,(u)\;du$$

where $c = u\,(a)$ and $d = u\,(b)$. u is the **substitution variable**.

To decipher this will probably require some examples. The general idea is to spot something in the integrand that is the derivative of something else (the substitution variable u) in the integrand.

Obvious Substitutions

Example 3

Find $\int x\, e^{x^2+4}\;dx$.

Note that differentiating $x^2 + 4$ gives $2x$, which is, apart from a factor of 2, equal to x. So, let $u\,(x) = x^2 + 4$. Then, $u\,'(x) = 2x$. $f\,(x) = e^x$ will give $f\,(u\,(x)) = e^{x^2+4}$. So, according to the Change of Variables Theorem,

$$\int 2x\, e^{x^2+4}\;dx = \int e^u\;du$$

$$\int x\, e^{x^2+4}\;dx = \frac{1}{2}\int e^u\;du$$

$$= \frac{1}{2}\,e^u + C$$

$$= \frac{1}{2}\,e^{x^2+4} + C$$

Normally, we don't explicitly use the detestable formula given in the theorem. In practice, we work with the differentials. So, in the example above, we would just write $du = 2x\,dx$, solve for $x\,dx$ and then go straight to the second line in the above calculation.

Example 4

Find $\displaystyle\int_1^2 \frac{2x + 3}{x^2 + 3x}\,dx \quad (x > 0)$.

Let $u = x^2 + 3x$. Then $du = (2x + 3)\,dx$. The new limits are $u(1) = 4$ and $u(2) = 10$. So,

$$\begin{aligned}\int_1^2 \frac{2x + 3}{x^2 + 3x}\,dx &= \int_4^{10} \frac{1}{u}\,du \\ &= \Big[\ln u\Big]_4^{10} \\ &= \ln(10) - \ln(4) \\ &= \ln\left(\frac{10}{4}\right) \\ &= \ln\left(\frac{5}{2}\right)\end{aligned}$$

Substitution examples involving definite integrals can be done in 2 ways. Like Example 4, the new limits are worked out and the integration is done with the new variable and the u limits are substituted in. Alternatively, the integration with the new variable is done and the answer written in terms of the old variable (just like indefinite integrals) and the old limits are substituted in. Which approach is better depends on the integral and personal preference.

$$\begin{aligned}\int_1^2 \frac{2x + 3}{x^2 + 3x}\,dx &= \int_{u(1)}^{u(2)} \frac{1}{u}\,du \\ &= \Big[\ln u\Big]_{u(1)}^{u(2)}\end{aligned}$$

$$= \left[\ln\left(x^2 + 3x\right)\right]_1^2$$

$$= \ln(10) - \ln(4)$$

$$= \ln\left(\frac{10}{4}\right)$$

$$= \ln\left(\frac{5}{2}\right)$$

There are lots of different types of substitution questions. Examples 3 and 4 are specific cases of 2 general types of 'easy' substitution.

- $\displaystyle\int (Df)\, f \; dx = \frac{1}{2}f^2 + C$

- $\displaystyle\int \frac{Df}{f} \; dx = \ln|f| + C$

Remember that Df means $f\,'$.

The difficulty in substitution questions is deciding which function to replace by u. Choose the one that differentiates to give something else (apart from possibly a numerical factor) in the integrand. In the following examples, try choosing u to be 'the other function' to see what goes wrong.

<u>Example 5</u>

Evaluate $\displaystyle\int_0^{\pi} \sin x \; e^{\cos x} \; dx$

Note that $D(\cos x) = -\sin x$. So, let $u = \cos x$, Then $du = -\sin x\, dx$, so $\sin x\, dx = -du$. Also, $u(0) = 1$ and $u(\pi) = -1$. So,

$$\int_0^{\pi} \sin x \; e^{\cos x} \; dx = -\int_1^{-1} e^u \; du$$

$$= \int_{-1}^{1} e^{u} \, du$$

$$= \left[e^{u} \right]_{-1}^{1}$$

$$= (e^{1}) - (e^{-1})$$

$$= e - \frac{1}{e}$$

Example 6

Integrate $x^{13} \, (2x^{14} - 9)^{-11}$.

Let $u = 2x^{14} - 9$. Then $du = 28x^{13} \, dx$. Thus,

$$\int x^{13} \, (2x^{14} - 9)^{-11} \, dx = \frac{1}{28} \int u^{-11} \, du$$

$$= \frac{1}{28} \left(\frac{u^{-10}}{-10} \right) + C$$

$$= -\frac{1}{280} (2x^{14} - 9)^{-10} + C$$

Example 7

Integrate $x^{73} \, \text{cosec}^{2} \, x^{74}$.

Let $u = x^{74}$. Then $du = 74x^{73} \, dx$. Hence,

$$\int x^{73} \, \text{cosec}^{2} \, x^{74} \, dx = \frac{1}{74} \int \text{cosec}^{2} \, u \, du$$

$$= \frac{1}{74} \, (-\cot u) + C$$

$$= -\frac{1}{74} \cot x^{74} + C$$

Example 8

Evaluate $\int_0^{\pi/2} \cos x \, \sin^{33} x \, dx$.

Let $u = \sin x$. Then $du = \cos x \, dx$. In this definite integral, we will write the answer in terms of x and use the x limits (try it the other way). We have,

$$\int_0^{\pi/2} \cos x \, \sin^{33} x \, dx = \int_{u_1}^{u_2} u^{33} \, du$$

$$= \left[\frac{u^{34}}{34} \right]_{u_1}^{u_2}$$

$$= \left[\frac{\sin^{34} x}{34} \right]_0^{\pi/2}$$

$$= \frac{1}{34}$$

Example 9

Find $\int \frac{\cos \sqrt{x}}{\sqrt{x}} \, dx$.

Let $u = \sqrt{x}$. Then $du = \frac{1}{2\sqrt{x}} \, dx$, so that $\frac{1}{\sqrt{x}} \, dx = 2 \, du$. So,

$$\int \frac{\cos \sqrt{x}}{\sqrt{x}} \, dx = 2 \int \cos u \, du$$

$$= 2 (\sin u) + C$$

$$= 2 \sin \sqrt{x} + C$$

Difficult Substitutions - Rewriting the Integrand

More difficult questions involve rewriting the integrand before or after the substitution phase. Normally, the substitution is given.

Example 10

Integrate $\cot x$ using the substitution $u = \sin x$.

Recalling the definition of $\cot x$,

$$\int \cot x \;\; dx = \int \frac{\cos x}{\sin x} \;\; dx$$

Given $u = \sin x$, $D(\sin x) = \cos x$. Then $du = \cos x \; dx$. So,

$$\int \cot x \;\; dx = \int \frac{du}{u}$$

$$= \ln |u| + C$$

$$= \ln |\sin x| + C$$

Note the alternative way of writing $\int \frac{1}{u} \; du$ in the first line.

Example 11

Integrate $\cos^5 x$ using the substitution $u = \sin x$.

Note that $\cos^5 x = \cos x \,.\, \cos^4 x = \cos x \; (\cos^2 x)^2 = \cos x \; (1 - \sin^2 x)^2 = \cos x \,(1 - 2\sin^2 x + \sin^4 x)$. Using the abbreviations C for $\cos x$ and S for $\sin x$, the integral becomes,

$$\int \cos^5 x \;\; dx = \int C\,(1 - 2S^2 + S^4) \;\; dx$$

Let $u = \sin x$. Then $du = \cos x \; dx$. Hence,

$$\int \cos^5 x \; dx = \int (1 - 2u^2 + u^4) \; du$$

$$= u - \frac{2u^3}{3} + \frac{u^5}{5} + C$$

$$= \sin x - \frac{2\sin^3 x}{3} + \frac{\sin^5 x}{5} + C$$

Example 12

Evaluate $\displaystyle\int_0^4 \frac{\sqrt{x}}{2 + \sqrt{x}} \, dx$ using the substitution $u = \sqrt{x} + 2$.

$u = \sqrt{x} + 2 \Rightarrow du = \dfrac{1}{2\sqrt{x}} dx \Rightarrow dx = 2\sqrt{x} \; du \Rightarrow \sqrt{x}\, dx = 2x$ $du \Rightarrow \sqrt{x}\, dx = 2(u - 2)^2 \; du$. Also, $u(0) = 2$ and $u(4) = 4$. Thus,

$$\int_0^4 \frac{\sqrt{x}}{2 + \sqrt{x}} \; dx = 2 \int_2^4 \frac{(u - 2)^2}{u} \; du$$

$$= 2 \int_2^4 \frac{(u^2 - 4u + 4)}{u} \; du$$

$$= \int_2^4 \left(2u - 8 + \frac{8}{u} \right) \; du$$

$$= \left[u^2 - 8u + 8\ln u \right]_2^4$$

$$= 8 \ln 2 - 4$$

Fill in the details for the last equality.

Areas

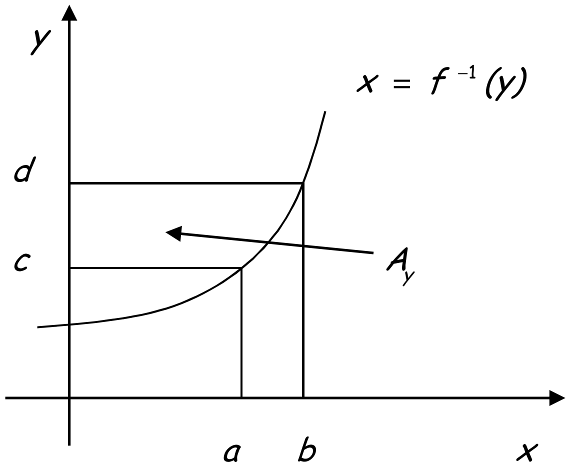
Recall that the area bounded by a curve, the lines $x = a$ and $x = b$ and the x – axis is given by,

$$A_x = \int_a^b f(x)\ dx$$

where the curve lies above the x – axis and $a < b$.

Area between Curve and the y – axis

The area bounded by a curve, and the y – axis can also be calculated.



Theorem:

The **area bounded by the curve $x = f^{-1}(y)$, the lines $y = c$ and $y = d$ ($c < d$) and the y – axis** where the curve lies to the right of the y – axis is given by,

$$A_y = \int_c^d f^{-1}(y)\ dy$$

Example 13

Find the area bounded by the curve $y = x^2$, the y – axis and the lines $y = 1$ and $y = 4$.

Since the y – values are both positive, the inverse function we are interested in is $x = \sqrt{y}$. Hence,

$$A_y = \int_1^4 y^{1/2} \; dy$$

$$= \left[\frac{2y^{3/2}}{3} \right]_1^4$$

$$= \left(\frac{2.4.\sqrt{4}}{3} \right) - \left(\frac{2.1.\sqrt{1}}{3} \right)$$

$$= \frac{14}{3}$$

Area between 2 Curves and the y – axis

The area between 2 curves and the y – axis can be found by a similar procedure to that between the x – axis. Just remember to invert the functions and integrate 'right-hand function minus left-hand function'.

Theorem:

The **area between 2 curves $x = f^{-1}(y)$ and $x = g^{-1}(y)$ ($f^{-1}(y) \geq g^{-1}(y)$), the y – axis and the lines $y = c$ and $y = d$ ($c < d$)** is given by,

$$A_y = \int_c^d \left(f^{-1}(y) - g^{-1}(y) \right) dy = \int_c^d f^{-1}(y) \; dy - \int_c^d g^{-1}(y) \; dy$$

Example 14

Fin the area of the region bounded by the curves $y = x^2$ and $y = 2x$ and the y – axis.

The functions meet when $x = 0$ and $x = 2$, or, in terms of y – values, $y = 0$ and $y = 4$. The inverted functions are $x = \frac{1}{2}y$ and $x = \sqrt{y}$. A quick sketch shows that $x = \sqrt{y}$ is the right-hand function. So,

$$A_y = \int_0^4 \left(y^{1/2} - \frac{y}{2} \right) dy$$

$$= \left[\frac{2y^{3/2}}{3} - \frac{y^2}{4} \right]_0^4$$

$$= \left(\frac{2.4.\sqrt{4}}{3} - \frac{4^2}{4} \right) - 0$$

$$= \frac{4}{3}$$

Check that the same answer is obtained by integrating between the x – axis.

Volumes of Solid of Revolution

Integration is also used to calculate volumes of regions.

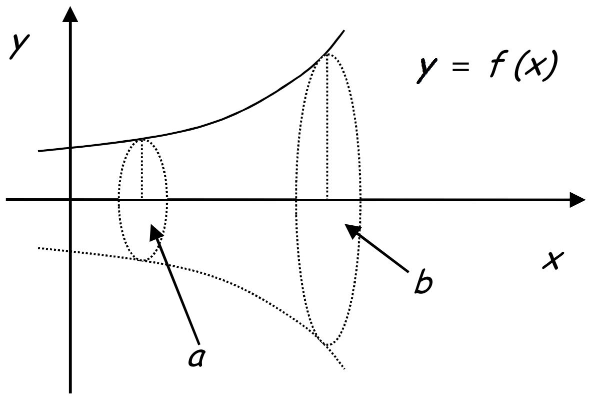
Definition:

A **solid of revolution** is a 3D shape formed by rotating a curve 360° about either the x – axis or y – axis.

Imagine part of the graph of a function rotated around either the x – axis or y – axis. The resulting figure is the shape whose volume we want to find. This volume is called the **volume of solid of revolution**. This technique can be used to derive formulae such as the volume of a cone, sphere and lots of other funky 3D shapes.

There are 2 cases to consider.

Volume Generated by a Curve about the x-axis



The diagram shows the graph of $y = f(x)$ rotated 360° about the x – axis between $x = a$ and $x = b$. The volume required is the region between the 2 dotted vertical lines and the 2 circles (which look like ellipses above due to perspective).

Theorem:

The **volume of solid of revolution generated by rotating the curve $y = f(x)$ 360° about the x – axis between $x = a$ and $x = b$ ($a < b$)** is given by,

$$V_x = \pi \int_a^b y^2 \, dx$$

Example 15

Find the volume of solid of revolution between the lines $x = 1$ and $x = 3$ when $y = x^2$ is rotated 360° about the x – axis.

$y = x^2 \Rightarrow y^2 = x^4$. So,

$$V_x = \pi \int_1^3 x^4 \, dx$$

$$= \pi \left[\frac{x^5}{5} \right]_1^3$$

$$= \frac{\pi}{5}\left(3^5 - 1^5\right)$$

$$= \frac{\pi}{5}(243 - 1)$$

$$= \frac{242\,\pi}{5}$$

Volume Generated by 2 Curves about the x-axis

The volume of a region formed between 2 curves by rotating them about the x – axis can be found by a similar technique to that in the previous theorem.

Theorem:

The **volume of solid of revolution formed between 2 curves y_1 and y_2 ($y_1 < y_2$) by rotating them 360° about the x – axis** is given by,

$$V_x = \pi \int_a^b \left(y_2^2 - y_1^2\right)\,dx$$

where the curves intersect at $x = a$ and $x = b$ ($a < b$).

Example 16

Find the volume of solid of revolution obtained by rotating the curves $y = x^2$ and $y^2 = 64x$ 360° about the x – axis.

The curves meet when $x^4 = 64x \Rightarrow x(x^3 - 64) = 0 \Rightarrow x = 0$ and $x = 4$. The 'top' curve is $y^2 = 64x$. So,

$$V_x = \pi \int_0^4 \left(64x - x^4\right)\,dx$$

$$= \pi \left[32x^2 - \frac{x^5}{5} \right]_0^4$$

$$= \pi \left(32.16 - \frac{4^5}{5} \right) - 0$$

$$= \pi \left(512 - \frac{1\,024}{5} \right)$$

$$= \frac{\pi}{5} \left(2\,560 - 1\,024 \right)$$

$$= \frac{1\,536\,\pi}{5}$$

Volumes about the y – axis are calculated in a similar manner to that about the x – axis.

Volume Generated by a Curve about the y-axis

<u>Theorem:</u>

The **volume of solid of revolution generated by rotating the curve $x = f^{-1}(y)$ 360° about the y – axis between $y = c$ and $y = d$ ($c < d$)** is given by,

$$V_y = \pi \int_c^d x^2 \, dy$$

<u>Example 17</u>

Calculate the volume of solid of revolution generated between $y = 1$ and $y = 9$ by rotating the curve $y = x^2$ 360° about the y – axis.

$$V_y = \pi \int_1^9 y \, dy$$

$$= \pi \left[\frac{y^2}{2} \right]_1^9$$

$$= 40\,\pi$$

Volume Generated by 2 Curves about the y-axis

Theorem:

The **volume of solid of revolution formed between 2 curves x_1 and x_2 ($x_1 < x_2$) by rotating them 360° about the y – axis** is given by,

$$V_y = \pi \int_c^d \left(x_2^{\,2} - x_1^{\,2} \right) dy$$

where the curves intersect at $y = c$ and $y = d$ ($c < d$).

Example 18

Calculate the volume of solid of revolution obtained by rotating the curves $y = x^2$ and $y^2 = 64x$ 360° about the y – axis.

From Example 16, the curves meet at $x = 0$ and $x = 4$, equivalently, $y = 0$ and $y = 16$. The 'right-hand' curve is $y = x^2$. Hence,

$$V_y = \pi \int_0^{16} \left(y - \frac{y^4}{4\,096} \right) dy$$

$$= \pi \left[\frac{y^2}{2} - \frac{y^5}{20\,480} \right]_0^{16}$$

$$= \pi \left(\frac{16^2}{2} - \frac{16^5}{20\,480} \right) - 0$$

$$= \pi \left(128 - \frac{256}{5} \right)$$

$$= \frac{\pi}{5}(640 - 256)$$

$$= \frac{384\pi}{5}$$

Rectilinear Motion

The formulae for acceleration and velocity lead to formulae for velocity and displacement respectively (by thinking of integration as the opposite of differentiation).

- $v(t) = \int a(t)\ dt$
- $s(t) = \int v(t)\ dt$

The whole story can by summarized by the following diagram.

$$s(t) \underset{\int}{\overset{\frac{d}{dx}}{\rightleftarrows}} v(t) \underset{\int}{\overset{\frac{d}{dx}}{\rightleftarrows}} a(t)$$

<u>Definition:</u>

A particle is **at rest** when it has 0 velocity.

A particle is **initially at rest** when it has 0 velocity at time 0.

<u>Example 19</u>

A particle initially at rest has acceleration described by the formula $a(t) = 3t^2 + 4t - 1$. If it initially has zero displacement, calculate the displacement, velocity and acceleration after 2 seconds.

$a(2) = 3(2)^2 + 4(2) - 1 \Rightarrow a(2) = 19$ m/s^2. The velocity is,

$$v(t) = \int \left(3t^2 + 4t - 1\right)\ dt$$

$$v(t) = t^3 + 2t^2 - t + C$$

The constant of integration is crucial here. When $t = 0$, $v = 0$. Hence, $C = 0$. So,

$$v(t) = t^3 + 2t^2 - t$$

$v(2) = (2)^3 + 2(2)^2 - (2) \Rightarrow v(2) = 14$ m/s. The displacement is,

$$s(t) = \int \left(t^3 + 2t^2 - t\right) \, dt$$

$$s(t) = \frac{t^4}{4} + \frac{2t^3}{3} - \frac{t^2}{2} + D$$

When $t = 0$, $s = 0$, Hence, $D = 0$. Thus,

$$s(t) = \frac{t^4}{4} + \frac{2t^3}{3} - \frac{t^2}{2}$$

$$s(2) = \frac{(2)^4}{4} + \frac{2(2)^3}{3} - \frac{(2)^2}{2} \Rightarrow s(2) = \frac{58}{3} \text{ m.}$$