Partial Fractions

Prerequisites: Solving simple equations; comparing coefficients; factorising simple quadratics and cubics; polynomial division.

Maths Applications: Integration; graph sketching.

Real-World Applications: Population growth.

Rational Functions

Definition:

A polynomial divided by another polynomial is called a **rational function**,

$$\frac{p(x)}{q(x)}$$

If deg $p <$ deg q, the above is called a **proper rational function**, whereas if deg $p \geq$ deg q, the above is called an **improper rational function**.

A rational function can be written in terms of a proper rational function.

Polynomial Long Division

Polynomial long division is a technique used to write an improper rational function in terms of a proper rational function. The technique can be illustrated by analogy with long division of whole numbers.

Example 1

Divide 3 692 by 15.

$$\begin{array}{r|l} & \ \ 2 \\ \hline 15 & 3692 \\ & 3000 \\ \hline & \ \ 692 \end{array}$$

$$3692 = (15 \times 200) + 692$$

Notice the 2 is in the hundreds column so it's really 200.

692 is not less than 15, so we continue,

$$\begin{array}{r|l} & \underline{\ \ 2\,4\ \ \ \ } \\ 15 & 3\,6\,9\,2 \\ & \underline{3\,0\,0\,0} \\ & \ \ 6\,9\,2 \\ & \underline{\ \ 6\,0\,0} \\ & \ \ \ \ 9\,2 \end{array} \qquad \begin{array}{l} 3692 = (15 \times 200) + 692 \\ \\ 692 = (15 \times 40) + 92 \end{array}$$

92 is not less than 15, so we continue,

$$\begin{array}{r|l} & \underline{\ \ 2\,4\,6\ \ } \\ 15 & 3\,6\,9\,2 \\ & \underline{3\,0\,0\,0} \\ & \ \ 6\,9\,2 \\ & \underline{\ \ 6\,0\,0} \\ & \ \ \ \ 9\,2 \\ & \underline{\ \ \ \ 9\,0} \\ & \ \ \ \ \ \ 2 \end{array} \qquad \begin{array}{l} 3\,692 = (15 \times 200) + 692 \\ \\ 692 = (15 \times 40) + 92 \\ \\ 92 = (15 \times 6) + 2 \end{array}$$

As 2 is less than 15, we stop.

Putting the 3 calculations on the RHS together gives,

$$3\,692 = (15 \times 200) + (15 \times 40) + (15 \times 6) + 2$$

In other words,

$$3\,692 = (15 \times 246) + 2$$

The quotient is 246 and the remainder is 2. This can be written in the alternative form,

$$\frac{3\,692}{15} = 246 + \frac{2}{15}$$

Example 2

Divide $2x^2 + 3x - 5$ by $x - 2$.

$$\begin{array}{r|l} & 2x \\ \hline x - 2 & 2x^2 + 3x - 5 \\ & 2x^2 - 4x \\ \hline & \quad\; 7x - 5 \end{array}$$

$$2x^2 + 3x - 5 = (x - 2)\,2x + 7x - 5$$

As $7x - 5$ has a degree which is not less than the degree of $x - 2$, we continue,

$$\begin{array}{r|l} & 2x + 7 \\ \hline x - 2 & 2x^2 + 3x - 5 \\ & 2x^2 - 4x \\ \hline & \quad\; 7x - 5 \\ & \quad\; 7x - 14 \\ \hline & \qquad\;\; + 9 \end{array}$$

$$2x^2 + 3x - 5 = (x - 2)\,2x + 7x - 5$$

$$7x - 5 = (x - 2)\,7 + 9$$

As the degree of 9 is less than the degree of $x - 2$, we stop.

Putting the two calculations on the RHS together gives, upon simplifying,

$$2x^2 + 3x - 5 = (x - 2)\,2x + (x - 2)\,7 + 9$$

$$2x^2 + 3x - 5 = (x - 2)(2x + 7) + 9$$

So, the quotient is $2x + 7$ and the remainder is 9. This can be written in the alternative form,

$$\frac{2x^2 + 3x - 5}{x - 2} = (2x + 7) + \frac{9}{x - 2}$$

Reducible and Irreducible polynomials

Some polynomials can be factorised using the real number system, others can't.

Definition:

A polynomial p is **reducible** if it can be factorised into polynomials, none of which are equal to the polynomials 1 and p.

A polynomial is **irreducible** if it is not reducible.

Partial Fraction Decompositions

Rational functions can be broken down into proper rational functions.

Theorem (**Partial Fraction Decomposition Theorem**):

Any rational function $\frac{p}{q}$ can be written as a polynomial plus a sum of proper rational functions each of which is of the form,

$$\frac{g(x)}{r(x)^n} \qquad (n \in \mathbb{N})$$

where r is an irreducible factor of q and $\deg g < \deg r$; such proper rational functions are called **partial fractions** of $\frac{p}{q}$.

Rational functions that have at most a cubic denominator will be studied. The cubic will factorise into either 3 linear factors (2 of which may be the same) or 1 linear and 1 irreducible quadratic factor.

Types of Partial Fractions Arising from a Cubic Denominator

In the following table, $a \neq 0$, each factor is non-zero and $S, T \in \mathbb{R}$.

Factor	Partial Fraction
$ax + b$ (**non-repeated linear**)	$\frac{S}{ax+b}$
$(ax+b)^2$ (**repeated linear**)	$\frac{S}{ax+b} + \frac{T}{(ax+b)^2}$
$ax^2 + bx + c$ (**irreducible quadratic**)	$\frac{Sx+T}{ax^2+bx+c}$

Long division can be done by dividing with a quadratic, cubic etc. For example, check (by expanding the denominator and then long dividing) that,

$$\frac{2x^2 + 3x - 4}{(x - 1)(x + 2)} = 2 + \frac{x}{(x - 1)(x + 2)}$$

3 Distinct Linear Factors

<u>Example 3</u>

Find partial fractions for $\dfrac{3x + 5}{(x + 1)(x + 2)(x - 3)}$.

$$\frac{3x + 5}{(x + 1)(x + 2)(x - 3)} = \frac{S}{x + 1} + \frac{T}{x + 2} + \frac{U}{x - 3}$$

Multiply both sides of the above equation by $(x + 1)(x + 2)(x - 3)$ to give,

$$3x + 5 = S(x + 2)(x - 3) + T(x + 1)(x - 3) + U(x + 1)(x + 2)$$

$x = -1$ gives,

$$3(-1) + 5 = S(-1 + 2)(-1 - 3) + 0 + 0$$

$$2 = S(1)(-4)$$

$$S = -\frac{1}{2}$$

Similarly, $x = -2$ and $x = 3$ respectively give,

$$T = -\frac{1}{5},\ U = \frac{7}{10}$$

Hence,

$$\frac{3x + 5}{(x + 1)(x + 2)(x - 3)} = -\frac{1}{2(x + 1)} - \frac{1}{5(x + 2)} + \frac{7}{10(x - 3)}$$

2 Distinct Linear Factors (with 1 of these Repeated)

<u>Example 4</u>

Express $\dfrac{x^2 + 6x - 4}{(x + 2)^2(x - 4)}$ in partial fractions.

$$\frac{x^2 + 6x - 4}{(x + 2)^2(x - 4)} = \frac{S}{x + 2} + \frac{T}{(x + 2)^2} + \frac{U}{x - 4}$$

$$x^2 + 6x - 4 = S(x + 2)(x - 4) + T(x - 4) + U(x + 2)^2$$

$x = -2$ gives,

$$4 - 12 - 4 = 0 + T(-6) + 0$$

$$6T = 12$$

$$T = 2$$

$x = 4$ gives,

$$16 + 24 - 4 = 0 + 0 + U(36)$$

$$36U = 36$$

$$U = 1$$

Comparing coefficients of x^2 gives,

$$1 = \mathrm{S} + \mathrm{U} \Rightarrow S = 0$$

So,

$$\frac{x^2 + 6x - 4}{(x + 2)^2(x - 4)} = \frac{2}{(x + 2)^2} + \frac{1}{x - 4}$$

1 Linear Factor and 1 Irreducible Quadratic Factor

Example 5

Find partial fractions for $\dfrac{x^2 - 4}{(3x + 2)(x^2 + 1)}$.

$$\frac{x^2 - 4}{(3x + 2)(x^2 + 1)} = \frac{S}{3x + 2} + \frac{Tx + U}{x^2 + 1}$$

$$x^2 - 4 = S(x^2 + 1) + (Tx + U)(3x + 2)$$

$x = -\dfrac{2}{3}$ gives,

$$\frac{4}{9} - 4 = S\left(\frac{4}{9} + 1\right) + 0$$

$$\frac{13}{9}S = -\frac{32}{9}$$

$$S = -\frac{32}{13}$$

Expanding out the RHS of the main equation gives,

$$x^2 - 4 = (S + 3T)x^2 + (2T + 3U)x + (S + 2U)$$

Comparing coefficients of both sides gives,

$$S + 3T = 1 \Rightarrow T = \frac{15}{13}$$

$$S + 2U = -4 \Rightarrow U = -\frac{10}{13}$$

Hence,

$$\frac{x^2 - 4}{(3x + 2)(x^2 + 1)} = -\frac{32}{13(3x + 2)} + \frac{15x - 10}{13(x^2 + 1)}$$