Properties of Functions

Prerequisites: Definition of a function, composing 2 functions; criterion for inverse function to exist; sketching inverses; finding extrema; sketching polynomial graphs; transformations of functions; increasing/decreasing functions; finding intercepts with axes.

Maths Applications: Graph sketching.

Real-World Applications: Modelling behaviour of systems.

The Modulus Function

Definition:

The **modulus** (aka **absolute value**) **of a number *a*** is the number denoted by $|a|$ and defined by,

$$|a| = \begin{cases} a & (a \geq 0) \\ -a & (a < 0) \end{cases}$$

For example, $|7| = 7$, $|0| = 0$ and $|-3| = 3$. This can be extended to functions.

Definition:

The **modulus** (aka **absolute value**) **of a function *f*** is the function denoted by $|f|$ and defined by,

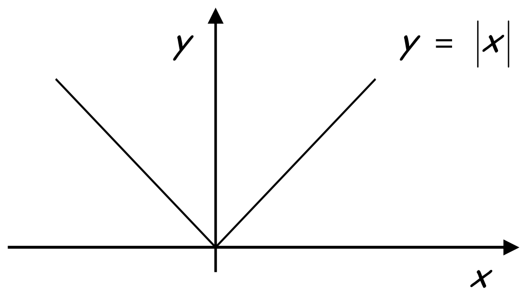
$$|f| = \begin{cases} f & (f \geq 0) \\ -f & (f < 0) \end{cases}$$

The modulus of a function takes the graph of f and makes any negative bits positive. The term *modulus function* is usually reserved for the function in the next example.

Example 1

Sketch the graph of $f(x) = |x|$.

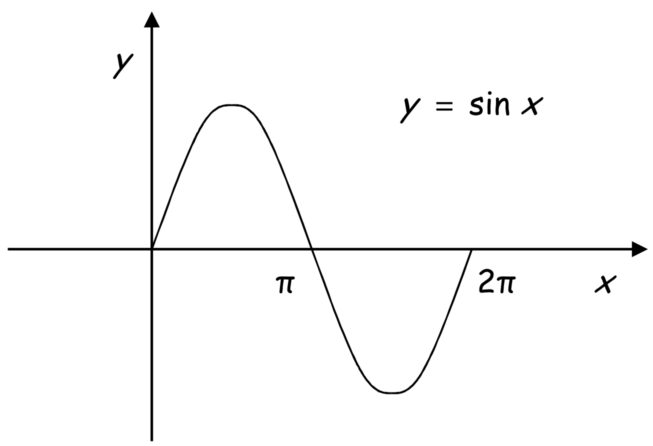
This is the graph of $f(x) = x$ reflected in the x – axis.



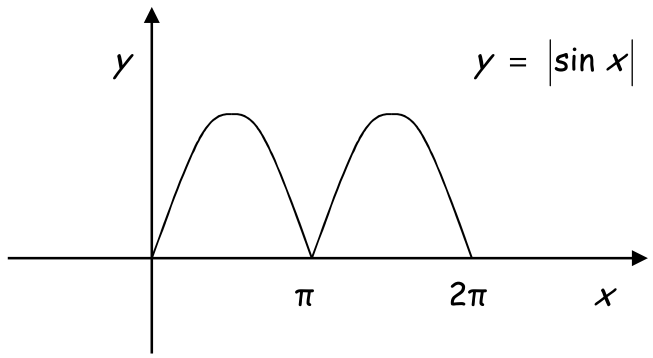
Example 2

Given that $f(x) = \sin x$, sketch the graph of $|f|$ in the interval $[0, 2\pi]$.

First sketch $\sin x$ in the given interval.

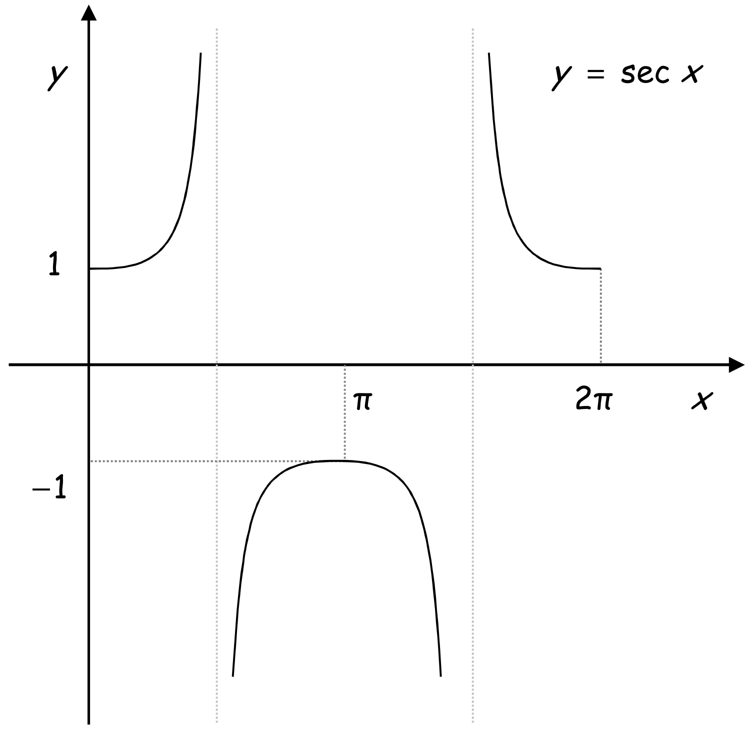


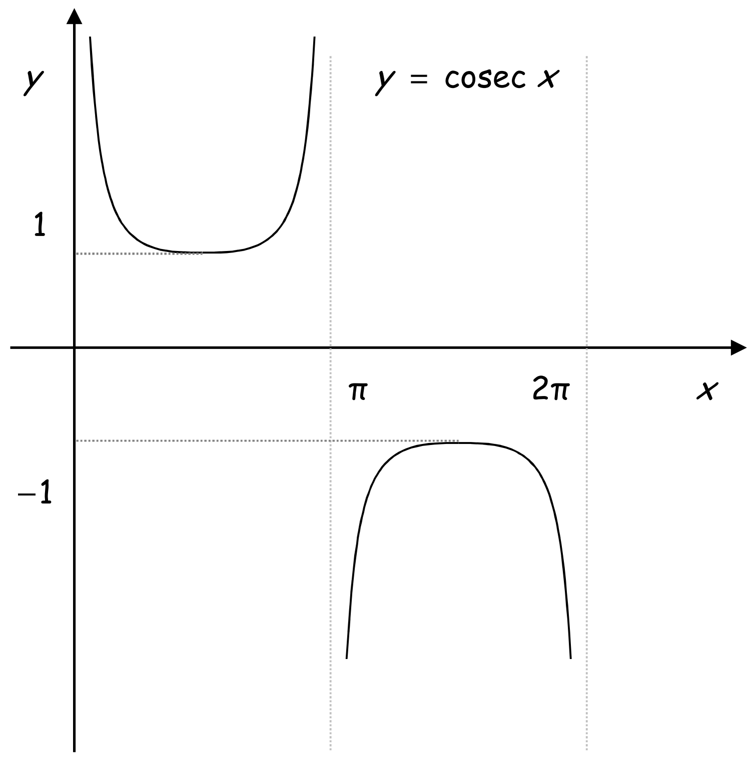
Then reflect the negative part of the graph in the x – axis.

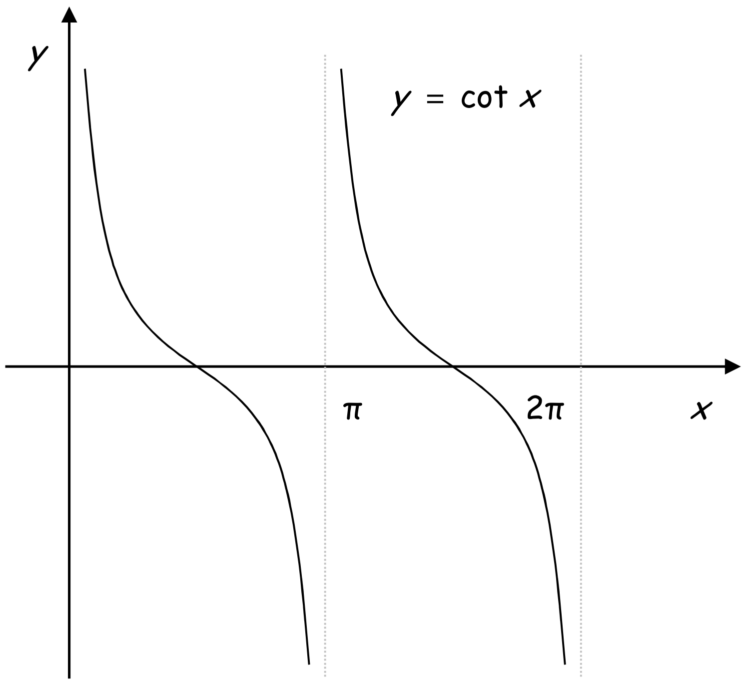


Graphs of Reciprocal Trigonometric Functions

The graphs of sec *x*, cosec *x* and cot *x* from 0 to 2π (apart from the few values where they are not defined) are shown below.







Inverse Functions

Recall that the graph of an inverse function can be sketched by reflecting the graph of the original function in the line $y = x$.

Finding a Formula for the Inverse Function

To find the inverse function, take the original function for y in terms of x, interchange x and y (reason: the input for the inverse will be the output for the original function), and solve this interchanged formula for y (make y the subject). This will be the formula for the inverse. The domain and range of the original function must be checked for the inverse to exist. There must be only a single x - value that gives a single y – value. Sketching the graph of the original function makes it obvious to determine when the inverse exists.

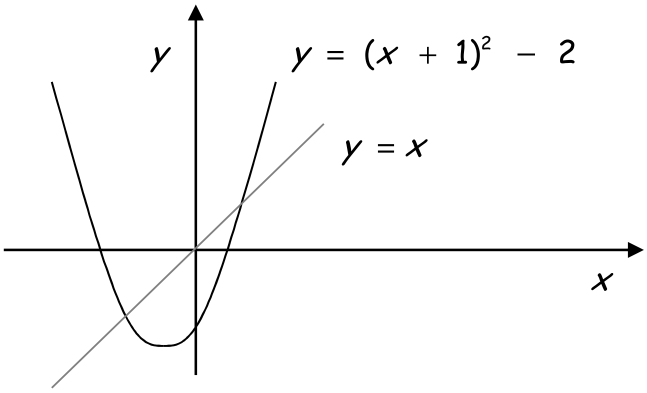
Example 3

Find the inverse of the function $f(x) = x^2 + 2x - 1$, stating the domain and range of f^{-1}.

Completing the square gives, $x^2 + 2x - 1 = (x + 1)^2 - 2$. So, the original function can be written as $y = (x + 1)^2 - 2$. Interchanging x and y gives $x = (y + 1)^2 - 2$. Solving this for y gives,

$$y = \pm\sqrt{x + 2} - 1$$

We appear to have 2 inverses ! The inverse of a function is unique, so what has gone wrong ? Sketch the graph of the original function to find the problem (and solution).



The problem is that one y – value has two x – values associated with it. The parabola has minimum turning point $(-1, -2)$. We have 2 choices for making one x – value match only one y – value. The demarcation point is $(-1, -2)$, so the choices are $(-\infty, -1]$ or $[-1, \infty)$. Restricting the domain of f to $[-1, \infty)$ (in English, $x \geq -1$) makes the range of f $[-2, \infty)$. Hence, for this choice of domain, the inverse is,

$$f^{-1}(x) = \sqrt{x + 2} - 1$$

(Restricting the domain of f to $(-\infty, -1]$ would give the inverse as $f^{-1}(x) = -\sqrt{x + 2} - 1$). Also,

$$\text{dom } f^{-1} = [-2, \infty)$$

$$\text{ran } f^{-1} = [-1, \infty)$$

Inverse Trigonometric Functions

Inverses of the trigonometric functions sine, cosine and tangent can be defined by restricting the domains. Again, there are choices for restricting the domains.

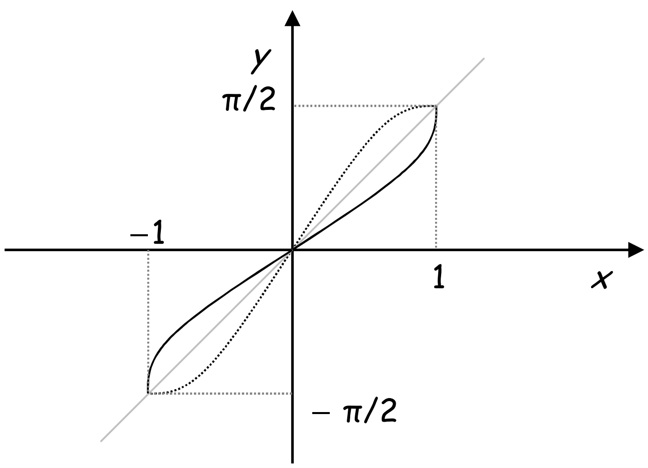
<u>Definition:</u>

The **inverse sine function** (aka **arcsine**), denoted by $\mathbf{sin^{-1}\ x}$, is defined as the inverse of the sine function. The domain and range of arcsine are,

$$\text{dom}\,(\sin^{-1} x) = [\,-1, 1]$$

$$\text{ran}\,(\sin^{-1} x) = \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$$

So, inverse sine is obtained by restricting the domain of sine to $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ and the range of sine to $[\,-1, 1]$. The graph of inverse sine and how it is derived from the restricted sine function is shown in the following diagram. All numbers refer to the inverse sine function.



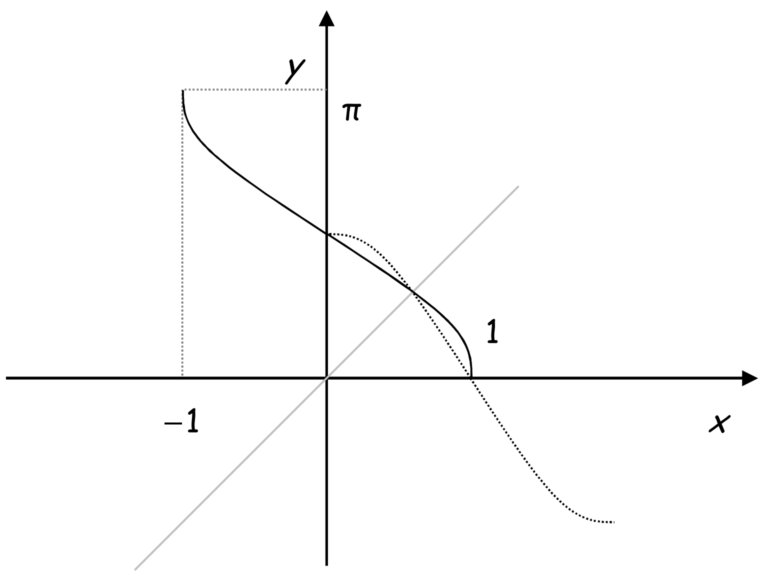
Definition:

The **inverse cosine** function (aka **arccosine**), denoted by $\cos^{-1} x$, is defined as the inverse of the cosine function. The domain and range of arccosine are,

$$\text{dom}\,(\cos^{-1} x) = [-1, 1]$$

$$\text{ran}\,(\cos^{-1} x) = [0, \pi]$$

So, inverse cosine is obtained by restricting the domain of cosine to $[0, \pi]$ and the range of cosine to $[-1, 1]$. The graph of inverse cosine and how it is derived from the restricted cosine function is shown in the following diagram. All numbers refer to the inverse cosine function.



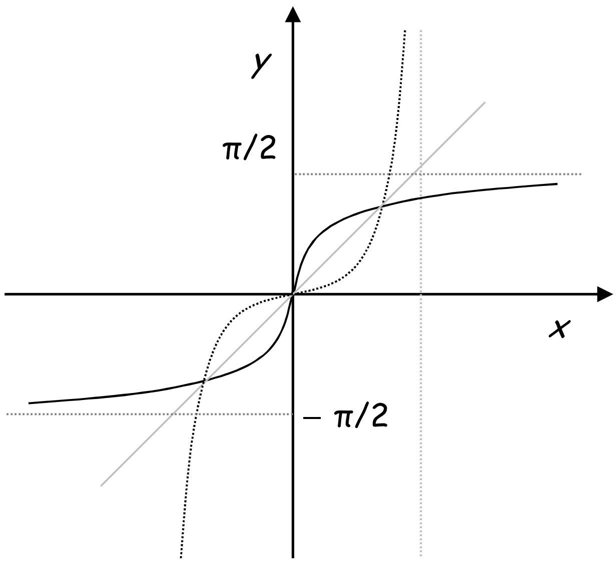
Definition:

The **inverse tangent** function (aka **arctan**), denoted by $\tan^{-1} x$ is defined as the inverse of the tangent function. The domain and range of arctan are,

$$\text{dom}\,(\tan^{-1} x) = \mathbb{R}$$

$$\text{ran}\,(\tan^{-1} x) = \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$$

So, inverse tangent is obtained by restricting the domain of tangent to $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ and the range of tangent to $\mathbb{R}$. The graphs of restricted tangent (dotted curve) and inverse tangent (solid curve) are shown in the following graph. All numbers refer to the inverse tangent function.



Warning: $\sin^{-1} x$ does not mean 'one divided by sin x' (that would be the reciprocal of sin x, i.e. cosec x) and similarly for $\cos^{-1} x$ and $\tan^{-1} x$. This is a very common mistake. The function $\sin^{-1}$ is the inverse of the sine function as defined above.

<u>Example 4</u>

State the domain and range of $\sin^{-1} 2x$.

Sketching the graph of $\sin 2x$ shows that the inverse can be defined when dom ($\sin 2x$) = $\left[-\frac{\pi}{4}, \frac{\pi}{4}\right]$ and ran ($\sin 2x$) = $[-1, 1]$. Hence, dom ($\sin^{-1} 2x$) = $[-1, 1]$ and ran ($\sin^{-1} 2x$) = $\left[-\frac{\pi}{4}, \frac{\pi}{4}\right]$.

Extrema and Inflexion Points Revisited

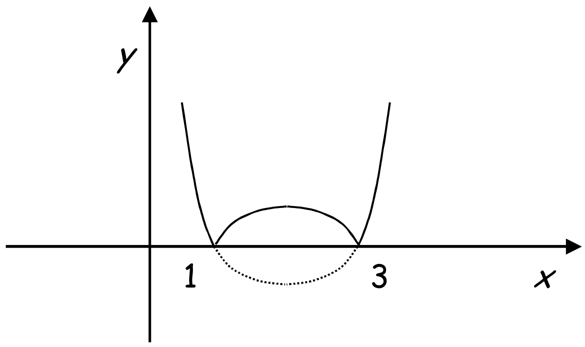
Extrema of Functions Involving the Modulus

Critical points often arise when the modulus of a function is taken, as sharp corners tend to be created when reflecting the negative parts of the graph of the function in the x – axis.

Example 5

Find the critical points of the function $f(x) = \left| x^2 - 4x + 3 \right|$ on $\mathbb{R}$ and classify them.

The graph looks as follows.

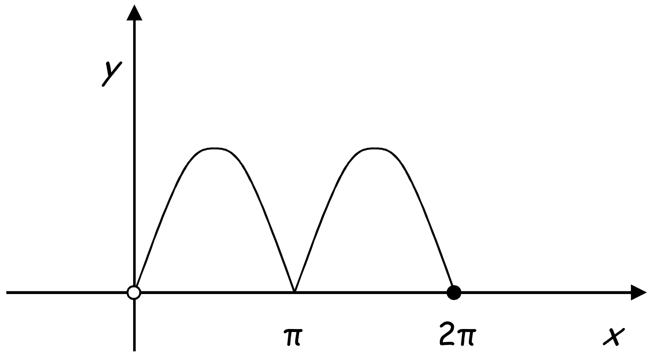


Clearly, the function has kinks at (1, 0) and (3, 0), so it is not differentiable at those points. It is differentiable everywhere else. The function $x^2 - 4x + 3$ has a (global) minimum at (2, -1); hence, $\left| x^2 - 4x + 3 \right|$ has a (local) maximum at (2, 1). So, the critical points of the given function are (1, 0) and (3, 0) (both local minima) and (2, 1) (local maximum).

Example 6

Find the extrema of the function $f(x) = \left| \sin x \right|$ on (0, 2π].

The relevant part of the graph from Example 2 is shown as follows.



Note that (0, 0) is not a critical point, as it is not in dom f. The critical points are the global maximum ($\pi/2$, 1), the global minimum (π, 0), the global maximum ($3\pi/2$, 1) and the endpoint minimum (2π, 0).

Inflexion Points of Trigonometric Functions

Example 7

Find the coordinates of the inflexion point of $f(x) = \cot x$ in $(0, \pi)$ and show that it is a non-horizontal inflexion point.

Differentiating gives $f'(x) = -\operatorname{cosec}^2 x$. As cosec x is never 0, there are clearly no maxima or minima. So, inflexion points will be found by solving $f''(x) = 0$. $f''(x) = 2\operatorname{cosec}^2 x \cot x$. Solving $f''(x) = 0$ gives $\cos x = 0$, and so $x = \pi/2$. Also, $f(\pi/2) = 0$ and $f'(\pi/2) = -1$. Hence, $(\pi/2, 0)$ is a non-horizontal inflexion point.

Example 8

Find the coordinates of the inflexion point of $f(x) = \tan x$ in the interval $(\pi/2, 3\pi/2)$ and determine the nature of the inflexion point and the gradient of the tangent line at the inflexion point.

Differentiating gives $f'(x) = \sec^2 x$. There are no solutions to $f'(x) = 0$, so there are no maxima or minima. Any inflexion points will be found by solving $f''(x) = 0$. $f''(x) = 2\sec^2 x \tan x$. Solving $f''(x) = 0$ gives $\sin x = 0$, and so $x = \pi$. Also, $f(\pi) = 0$ and $f'(\pi) = 1$. Hence, $(\pi, 0)$ is a non-horizontal inflexion point and the required gradient is 1.

Even and Odd Functions

Some functions have special types of symmetry. These can be easily identified from the graph of the function.

Definition:

A function f is **even** if $f(x) = f(-x)$ $(\forall x \in \text{dom } f)$.

Corollary:

An even function has a graph that is symmetrical about the y – axis.

Definition:

A function f is **odd** if $f(x) = -f(-x)$ $(\forall x \in \text{dom } f)$.

Corollary:

A function is odd if reflecting the graph of the function in the x – axis and then the y – axis (or the other way around) results in a graph that is identical with the original. This is the same as rotating the graph 180° about the origin.

Definition:

A function f is **neither (even nor odd)** if it is not even and not odd.

Corollary:

A function is neither if the graph of the function is not symmetrical about the y – axis or if the graph is not the same after a 180° rotation about the origin.

The corollaries give graphical determinations of deciding whether a function is even, odd or neither. In the case when a graph is not given, other approaches must be used.

Deciding whether a given function is even or odd involves obtaining $f(-x)$ and then comparing this with $f(x)$.

Verifying that a Function is Even

Example 9

Show that the function $f(x) = x^3 \sin x$ is even. Clearly, dom $f = \mathbb{R}$. Thus, for any $x \in \mathbb{R}$,

$$\begin{aligned} f(-x) &= (-x)^3 \sin(-x) \\ &= -x^3 (-\sin x) \\ &= x^3 \sin x \\ &= f(x) \end{aligned}$$

Hence, as $f(x) = f(-x)$ $(\forall x \in \text{dom } f)$, f is even.

Verifying that a Function is Odd

Example 10

Show that the function $f(x) = x - \dfrac{x^3}{\cos x}$ is odd. Note that

dom $f = \mathbb{R} \setminus \left\{ x \in \mathbb{R} : x = \dfrac{\pi}{2} + \pi n,\ n \in \mathbb{Z} \right\}$ (in English, all real numbers apart from odd multiples of π/2). Thus, for any $x \in$ dom f,

$$\begin{aligned} f(-x) &= (-x) - \frac{(-x)^3}{\cos(-x)} \\ &= -x - \frac{(-x^3)}{\cos x} \\ &= -x + \frac{x^3}{\cos x} \\ &= -\left(x - \frac{x^3}{\cos x} \right) \\ &= -f(x) \end{aligned}$$

Hence, as $f(x) = -f(-x)$ $(\forall x \in \text{dom } f)$, f is odd.

Verifying that a Function is Neither Even nor Odd

For the case of neither, a slightly different approach is required. If a function is not even, then it must be shown that at least 1 x - value makes the statement $f(x) = f(-x)$ false; if a function is not odd, then it must be shown that at least 1 x - value makes the statement $f(x) = -f(-x)$ false.

<u>Example 11</u>

Determine whether the function $f(x) = x^2 + e^{-x}$ is even, odd or neither.

$$f(-x) = (-x)^2 + e^{-(-x)}$$

$$f(-x) = x^2 + e^x$$

It looks as though f is neither even nor odd. But this requires justification. For example, $f(1) = 1 + e^{-1}$ and $f(-1) = 1 + e$. Clearly, $f(1) \neq f(-1)$ (as $e^{-1} \neq e$) and $f(1) \neq -f(-1)$ (as both are positive). Hence, f is neither.

<u>Asymptotes</u>

<u>Definition:</u>

An **asymptote** is a line (not necessarily straight), that a function approaches as the x – values approach a certain value.

Types of Asymptotes

There are 3 types of asymptotes. The first 2 types are always straight lines, but the third type could be straight or curved. In this course, all asymptotes will be straight lines.

Asymptotes occur where a function is not defined.

<u>Definition:</u>

A **vertical asymptote** is of the form $x =$ constant.

<u>Corollary:</u>

A function f has as vertical asymptote the straight line $x = a$ if either $\lim_{x \to a^-} f(x) = \pm \infty$ or $\lim_{x \to a^+} f(x) = \pm \infty$ (both statements may be true).

A vertical asymptote is parallel to the y – axis.

<u>Definition:</u>

A **horizontal asymptote** is of the form $y =$ constant.

<u>Corollary:</u>

A function $f(x)$ has as horizontal asymptote the straight line $y = b$ if either $\lim_{x \to -\infty} f(x) = b$ or $\lim_{x \to +\infty} f(x) = b$.

A horizontal asymptote is parallel to the x – axis.

<u>Definition:</u>

An **oblique asymptote** is of the form $y = mx + c$ $(m \neq 0)$.

A similar corollary can be given for oblique asymptotes, but it's more useful to analyse each example in turn.

An oblique asymptote is a straight line that has non-zero gradient.

A quick way of deciding where asymptotes will occur for a given function is to spot x – values which make the following happen.

- Zero denominator.
- Negative square root.
- Logarithm of zero.

Asymptotes of Rational Functions

Recall that any rational function $\frac{p}{q}$ can be written (by long division, if required) as a polynomial f plus a proper rational function $\frac{g}{q}$, i.e. ,

$$\frac{p(x)}{q(x)} = f(x) + \frac{g(x)}{q(x)}$$

Normally, q will factorise further (the Partial Fraction Decomposition Theorem breaks down $\frac{g}{q}$ further into partial fractions). The vertical asymptotes are found by solving $q(x) = 0$. A horizontal asymptote occurs if f is a constant function, whereas an oblique asymptote occurs if f is a linear function with non-zero gradient.

Example 12

Find all the asymptotes of $p(x) = \frac{2x^2 + 3x - 4}{(x - 1)(x + 2)}$. Long division (need to expand out the denominator first !) gives,

$$p(x) = 2 + \frac{x}{(x - 1)(x + 2)}$$

Hence, p has horizontal asymptote $y = 2$ and vertical asymptotes $x = 1$ and $x = -2$.

Example 13

Find all the asymptotes of $b(x) = \frac{2x^2 - 3x + 4}{x - 1}$. Long division gives,

$$b(x) = (2x - 1) + \frac{3}{x - 1}$$

Hence, b has oblique asymptote $y = 2x - 1$ and vertical asymptote $x = 1$.

Asymptotes of Trigonometric Functions

<u>Example 14</u>

State all the asymptotes of $y = \sec 2x$ in the interval $[0, 2\pi]$.

There are 2 whole secant graphs in the interval $[0, 2\pi]$. $\sec x$ has 2 vertical asymptotes, at $\pi/2$ and $3\pi/2$. Hence, $\sec 2x$ has 4 vertical asymptotes, at $\pi/4$, $3\pi/4$, $5\pi/4$ and $7\pi/4$ (sketch the graph).

Sketching Rational Functions

Rational functions can be sketched by analysing the following features.

- Stationary points.
- Inflexion points.
- Asymptotes.
- Intercepts with axes.

<u>Example 15</u>

Sketch the graph of $y = f(x)$ where $f(x) = 2 + \dfrac{x}{(x - 1)(x + 1)}$.

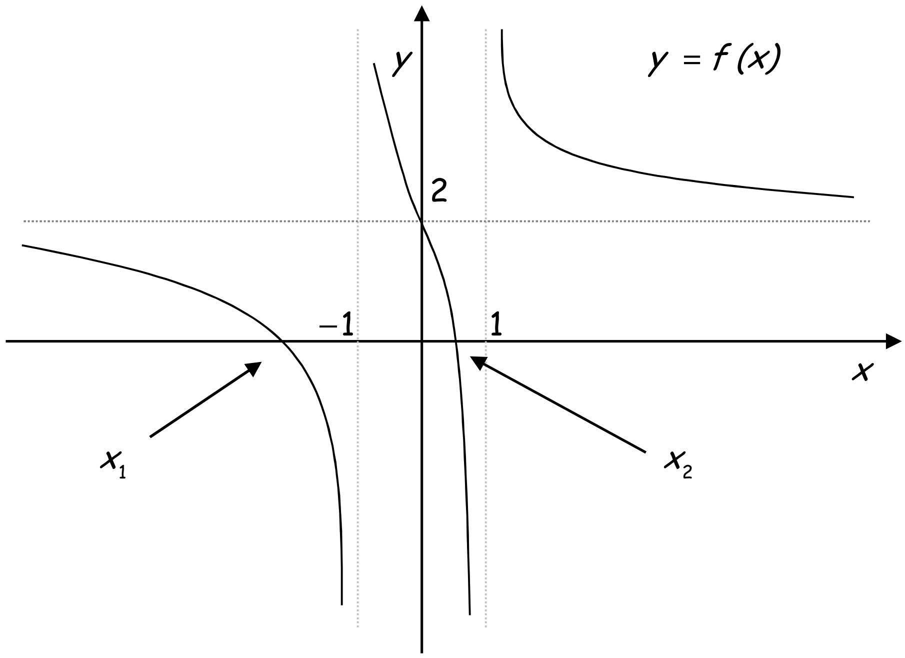
f has horizontal asymptote $y = 2$ and vertical asymptotes $x = 1$ and $x = -1$. As $x \to -1^-$, $y \to -\infty$ and as $x \to -1^+$, $y \to +\infty$ (pick values for x, for example, $x = -1 \cdot 1$ and experiment with a calculator; or just spot quickly that, with this value for x, the numerator is negative and the denominator is positive, while the whole fraction gets larger when this is repeated for x– values closer to -1). As $x \to 1^-$, $y \to -\infty$ and as $x \to 1^+$, $y \to +\infty$ When $x = 0$, $y = 2$. As $x \to -\infty$, $y \to 2^-$. As $x \to \infty$, $y \to 2^+$. When $y = 0$, the quadratic formula gives $x = \dfrac{-1 \pm \sqrt{17}}{4}$. Also,

$$f'(x) = -\frac{(x^2 + 1)}{(x^2 - 1)^2}$$

so f clearly has no stationary points. However,

$$f''(x) = \frac{2x(x^2 + 3)}{(x^2 - 1)^3}$$

does have one solution at $x = 0$. So, there is a point of inflexion at (0, 2). Also, $f'(0) = -1$, so (0, 2) is a non-horizontal (in fact, decreasing) inflexion point. We now have all the ingredients to sketch the graph.



$x_1 = \dfrac{-1 - \sqrt{17}}{4} \approx -1 \cdot 3$ and $x_2 = \dfrac{-1 + \sqrt{17}}{4} \approx 0 \cdot 8$. Don't worry if the graph isn't too accurate; remember, it's only a *sketch*, not a drawing.

<u>Example 16</u>

Sketch the graph of $y = f(x)$ where $f(x) = (x - 1) + \dfrac{1}{x - 2}$.

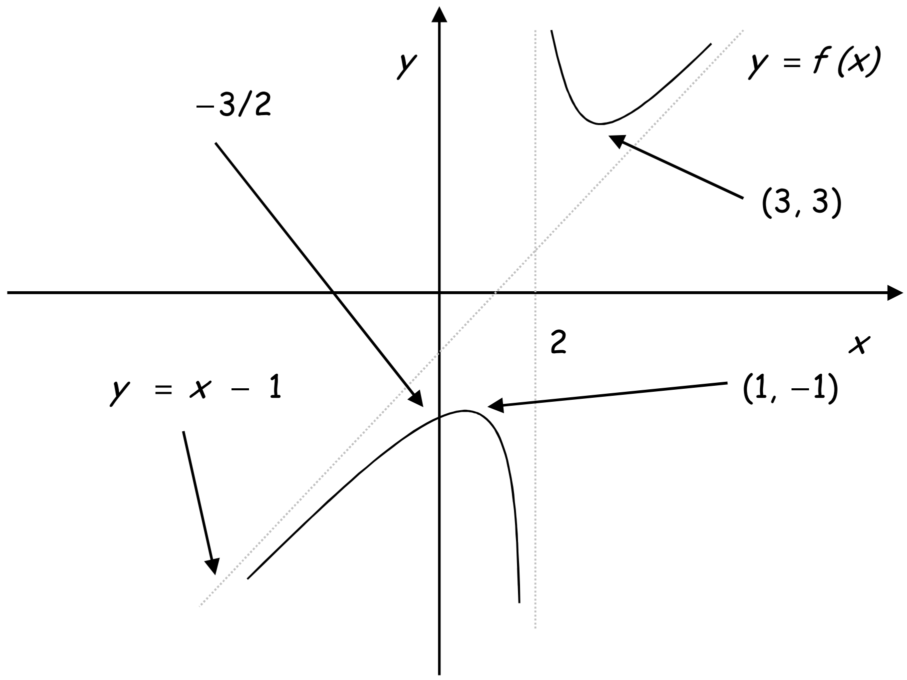
f has vertical asymptote $x = 2$ and horizontal asymptote $y = x - 1$. As $x \to 2^-$, $y \to -\infty$ and as $x \to 2^+$, $y \to +\infty$. As $x \to -\infty$, $y \to (x - 1)^-$. As $x \to \infty$, $y \to (x - 1)^+$. When $x = 0$, $y = -3/2$. When $y = 0$, any x–intercepts are given by solving the equation, $x^2 - 3x + 3 = 0$; but this quadratic has a negative discriminant, so there are no x– intercepts. Also,

$$f'(x) = 1 - \frac{1}{(x - 2)^2}$$

Solving the equation $f'(x) = 0$ gives $x = 1$ and $x = 3$. The second derivative is,

$$f''(x) = \frac{2}{(x - 2)^3}$$

$f''(1) = -2$, so, by the Second Derivative Test, $(1, -1)$ is a local maximum. Similarly, $f''(3) = 2$ shows that $(3, 3)$ is a local minimum. Clearly, $f''(x) = 0$, so there are no inflexion points.



Transformations of Functions

There are various types of functional transformations that must be known. Apart from taking inverses and the modulus, these transformations include reflections, translations and scales (as well as a combination of any of these 5 types of transformations).

When sketching transformed graphs, new points must be indicated if old points are given.

Sketching a Reflection of a Function in the x – axis

Each point (x, y) gets sent to $(x, -y)$: $f(x) \rightarrow -f(x)$.

Sketching a Reflection of a Function in the y – axis

Each point (x, y) gets sent to $(-x, y)$: $f(x) \rightarrow f(-x)$.

Sketching a Translation of a Function along the x – axis

Each point (x, y) gets sent to $(x + k, y)$; to the right if $k < 0$ and to the left if $k > 0$: $f(x) \rightarrow f(x + k)$.

Sketching a Translation of a Function along the y – axis

Each point (x, y) gets sent to $(x, y + k)$; up the way if $k > 0$ and down the way if $k < 0$: $f(x) \rightarrow f(x) + k$.

Sketching a Scaling of a Function along the x – axis

Each point (x, y) gets sent to (kx, y); stretching if $k < 1$ and squeezing if $k > 1$: $f(x) \rightarrow f(kx)$.

Sketching a Scaling of a Function along the y – axis

Each point (x, y) gets sent to (x, ky); stretching if $k > 1$ and squeezing if $k < 1$: $f(x) \rightarrow kf(x)$.

These 6 transformations should be familiar from Higher.

<u>Example 17</u>

Sketch the graph of $g(x) = \left|\dfrac{x^3}{x - 2}\right| + 1$.

First look at the simpler function $f(x) = \dfrac{x^3}{x - 2}$. This has,

$$f'(x) = \frac{2x^2(x - 3)}{(x - 2)^2}$$

and

$$f''(x) = \frac{2x(x^2 - 6x + 12)}{(x - 2)^3}$$

Check that there is a local minimum at (3, 27) and a horizontal point of inflexion at (0, 0) (for the inflexion point, use a table of signs or use the change of concavity criterion). There is a vertical asymptote at $x = 2$. The graph of f is indicated by a dotted line, whereas the graph of g is indicated by the solid line.