Sequences and Series

Prerequisites: Recurrence relations; solving linear and quadratic equations; solving simultaneous equations.

Maths Applications: Extending the Binomial Theorem; Maclaurin series.

Real-World Applications: Quantum mechanics.

Sequences and Series

Definition:

A **(real) sequence** is a function $f : \mathbb{N} \to \mathbb{R}$. The values of a sequence are traditionally denoted u_n (the ***n* th term**) , which clearly equals *f (n)*, whereas the sequence itself is denoted $\{u_n\}$.

A real sequence is just a list of real numbers in order. If $\mathbb{R}$ is replaced with $\mathbb{C}$, then we have a *complex sequence*. In this course, we will almost always deal with real sequences.

Example 1

1, 4, 9, 16, 25 . . . is a sequence. A function *f* which generates this sequence is, $f(n) = n^2$.

When adding the terms of a sequence, we can choose to add up some or all of the terms.

Series can thus be of 2 types: finite or infinite.

Definition:

A **finite series** is the sum of some terms of a sequence.

The terms of a sequence added up from 1st to n^{th} has a special name.

Definition:

The **sum to *n* terms** (aka **sum of the first *n* terms** or ***n* th partial sum**) of a sequence is,

$$S_n \stackrel{def}{=} \sum_{r=1}^{n} u_r$$

This definition is an example of a finite series (aka finite sum).

Corollary:

The n^{th} term of a sequence $\{u_n\}$ is given by,

$$u_{n+1} = S_{n+1} - S_n$$

Example 2

If the sum of the first 13 terms of a sequence is 37 and the sum of the first 14 terms is 39, find the value of u_{14}.

$$u_{14} = S_{14} - S_{13}$$

$$= 39 - 37$$

$$= 2$$

Definition:

An **infinite series** is the sum of all the terms of a sequence.

Definition:

The **sum to infinity** (aka **infinite sum**) of a sequence is the limit (if it exists) as $n \to \infty$ of the n^{th} partial sums, i.e.,

$$S_\infty \stackrel{def}{=} \lim_{n\to\infty} S_n$$

Definition:

An infinite series **converges** (aka **is summable**) if S_∞ exists; otherwise, the series **diverges** (aka **is not summable** aka **limit does not exist**).

Traditionally, the first term of a sequence is denoted by a. There are 2 important types of sequences we will study in depth. They are defined by recurrence relations.

Arithmetic Sequences and Series

Definition:

An **arithmetic sequence** is one in which the difference (aka **common difference d**) of any 2 successive terms is the same,

$$d \stackrel{def}{=} u_{n+1} - u_n$$

Example 3

Verify that 37, 26, 15, 4, -7, . . . is an arithmetic sequence.

We need to check that the difference between any 2 successive terms is the same. $u_2 - u_1 = -11$ and $u_3 - u_2 = -11$. Hence, as successive differences are the same, the sequence is an arithmetic sequence.

n^{th} term

Theorem:

The n^{th} term of an arithmetic sequence is given by,

$$u_n = a + (n - 1)\,d \qquad (a \in \mathbb{R}\ ,\, d \in \mathbb{R} \setminus \{0\})$$

If $d = 0$, then we end up with a constant sequence $a, a, a, \ldots$, which is not particularly interesting.

<u>Example 4</u>

Find a formula for the n^{th} term of the arithmetic sequence that starts 12, 19, 26, 33, 40, ...

The common difference is easily seen to be $d = 7$. The first term is 12. Hence,

$$u_n = 12 + (n - 1)\,7$$

$$u_n = 12 + 7n - 7$$

$$u_n = 7n + 5$$

<u>Example 5</u>

An arithmetic sequence has second term 4 and seventh term 19. Find a formula for the n^{th} term of this sequence.

We have,

$$u_2 = a + (2 - 1)\,d = 4$$

$$u_7 = a + (7 - 1)\,d = 19$$

which become,

$$a + d = 4$$

$$a + 6d = 19$$

Solving these simultaneous equations gives $d = 3$ and $a = 1$. Thus,

$$u_n = 1 + (n - 1)\,3$$

$$u_n = 3n - 2$$

<u>Example 6</u>

An arithmetic sequence has first term 6, common difference 3 and $u_n = 72$. Find the value of n.

$$72 = 6 + (n - 1)\,3$$

$$n - 1 = 22$$

$$n = 23$$

Example 7

An arithmetic sequence has first term -3 and $u_3 = 14$. Find the value of d.

$$14 = -3 + (3 - 1)\,d$$

$$d = \frac{17}{2}$$

Example 8

An arithmetic sequence has common difference 9 and $u_{16} = 68$. Find the value of a.

$$68 = a + (15)9$$

$$68 = a + 135$$

$$a = -67$$

Sum to n Terms

Definition:

The sum to n terms of an arithmetic sequence is given by,

$$S_n = \frac{n}{2}\left(2a + (n - 1)d\right)$$

It is clear that this sum is a quadratic in n.

Corollary:

The sum to n terms of an arithmetic sequence can always be written as,

$$S_n = P n^2 + Q n \quad (P \in \mathbb{R} \setminus \{0\}, Q \in \mathbb{R})$$

The definition gives the main formula to use, but the corollary can be useful too.

Example 9

Find S_{12} for the arithmetic sequence that starts 5, 8, 11, 14, 17, ...

For this sequence, $d = 3$ and $a = 5$. With $n = 12$, we have,

$$S_{12} = \frac{12}{2}(2(5) + (12 - 1)3)$$

$$= 6(10 + 33)$$

$$= 258$$

Example 10

An arithmetic sequence has first term 2 and common difference 3. Find the smallest value of n for which $S_n > 43$.

$$\frac{n}{2}(2(2) + (n - 1)3) > 43$$

$$n(3n + 1) > 86$$

$$3n^2 + n - 86 > 0$$

Solving the associated quadratic equation $3n^2 + n - 86 = 0$ gives $n = -5 \cdot 52\ldots$ and $n = 5 \cdot 19\ldots$. As $n \in \mathbb{N}$, this means that $n \geq 6$.

Example 11

An arithmetic sequence has first term 12 and $S_{14} = 238$. Find the common difference.

$$\frac{14}{2}\,(2(12) + (14 - 1)d) = 238$$

$$24 + 13d = 34$$

$$13d = 10$$

$$d = \frac{10}{13}$$

Example 12

An arithmetic sequence has common difference -8 and $S_8 = 16$. Find the first term.

$$\frac{8}{2}\,(2a + 6(-8)) = 16$$

$$2a - 48 = 4$$

$$a = 26$$

Clearly, adding the terms of an arithmetic sequence will make successive partial sums larger and larger in magnitude. This leads to the following.

Theorem:

The sum to infinity of an arithmetic sequence does not exist.

Some people say that the sum is infinite. Those people are not writing these notes.

Geometric Sequences and Series

Definition:

A **geometric sequence** is one in which the ratio (aka **common ratio** r) of any 2 successive terms is the same,

$$r \stackrel{def}{=} \frac{u_{n+1}}{u_n}$$

Example 13

Verify that 3, 6, 12, 24, 48, . . . is a geometric sequence.

We need to check that the ratio of any 2 successive terms is the same. $\frac{u_2}{u_1} = 2$ and $\frac{u_3}{u_2} = 2$. Hence, as successive ratios are the same, the sequence is.

n^{th} term

Theorem:

The n^{th} term of a geometric sequence is given by,

$$u_n = a\,r^{n-1} \qquad (a \in \mathbb{R} \setminus \{0\},\ r \in \mathbb{R} \setminus \{0, 1\})$$

If $a = 0$ or $r = 0$, then we end up with the trivial sequence 0, 0, 0, . . . , whereas if $r = 1$, we end up with a constant sequence a, a, a, . . . , neither of which are interesting.

Example 14

Find a formula for the n^{th} term of the geometric sequence that starts 400, 200, 100, 50, 25, ...

The common ratio is easily seen to be $r = \frac{1}{2}$. The first term is 400. Hence,

$$u_n = 400\left(\frac{1}{2}\right)^{n-1}$$

Example 15

A geometric sequence has third term 8 and fifth term 32. If the common ratio is negative, find a formula for the n^{th} term of this sequence.

We have,

$$u_3 = a\,r^2 = 8$$

$$u_3 = a\,r^4 = 32$$

Dividing the first equation by the second gives (and cancelling a, as it's non-zero)

$$r^2 = 4$$

As $r < 0$, $r = -2$. Substituting this back into either of the above 2 equations gives $a = 4$. The n^{th} term formula is thus,

$$u_n = 4\,.\,(-2)^{n-1}$$

Example 16

A geometric sequence has first term 2, common ratio 4 and $u_n = 128$. Find the value of n.

$$128 = 2\,.\,4^{n-1}$$

$$4^{n-1} = 64$$

$$4^{n-1} = 4^3$$

$$n = 4$$

<u>Example 17</u>

A geometric sequence has first term 10 and $u_3 = 5$. If $r > 0$, find the value of r.

$$10\,r^2 = 5$$

$$r^2 = \frac{1}{2}$$

$$r = \frac{1}{\sqrt{2}}$$

<u>Example 18</u>

A geometric sequence has common ratio 2 and $u_6 = 1\,024$. Find the value of a.

$$a\,.\,2^5 = 1\,024$$

$$32\,a = 1\,024$$

$$a = 32$$

Sum to n Terms

<u>Theorem:</u>

The sum to n terms of a geometric sequence is given by,

$$S_n = \frac{a\,(1 - r^n)}{1 - r}$$

Notice that the denominator won't be 0, as r cannot equal 1.

<u>Example 19</u>

Find S_8 for the sequence 6, 2, $\frac{2}{3}$, $\frac{2}{9}$, ...

The sequence is clearly a geometric one, with $a = 6$ and $r = \frac{1}{3}$. Hence,

$$S_8 = \frac{6\left(1 - \left(\frac{1}{3}\right)^n\right)}{2/3}$$

$$S_8 = 9\left(1 - \left(\frac{1}{3}\right)^n\right)$$

Example 20

A geometric sequence has first term 1 and common ratio 4. Find the smallest value of n for which $S_n > 2\,649$.

$$\frac{1 - 4^n}{1 - 4} > 2\,649$$

$$-\frac{1}{3}(1 - 4^n) > 2\,649$$

$$1 - 4^n < -7\,947$$

$$4^n > 7\,948$$

$$n.\ln 4 > \ln\ 7\,948$$

$$n > \frac{\ln 7\,948}{\ln 4}$$

$$n > 6 \cdot 471\,89$$

As $n \in \mathbb{N}$, $n = 7$.

Example 21

A geometric sequence has first term 7 and $S_2 = 6$. Find the common ratio.

$$\frac{7\,(1 - r^2)}{1 - r} = 6$$

$$\frac{7\,(1 - r)(1 + r)}{1 - r} = 6$$

As $r \neq 1$, we can cancel $(1 - r)$ to get,

$$7\,(1 + r) = 6$$

$$1 + r = \frac{6}{7}$$

$$r = -\frac{1}{7}$$

<u>Example 22</u>

A geometric sequence has common ratio $\frac{1}{5}$ and $S_3 = \frac{1}{25}$. Find the first term.

$$\frac{a\left(1 - \left(\frac{1}{5^3}\right)\right)}{4/5} = \frac{1}{25}$$

$$\frac{5a\left(1 - \frac{1}{125}\right)}{4} = \frac{1}{25}$$

$$\frac{5a\left(\frac{124}{125}\right)}{4} = \frac{1}{25}$$

$$\frac{31\,a}{25} = \frac{1}{25}$$

$$a = \frac{1}{31}$$

Sum to Infinity

Theorem:

The sum to infinity of a geometric sequence exists when $|r| < 1$ and is given by,

$$S_\infty = \frac{a}{1 - r}$$

Example 23

Determine whether the geometric sequence $1, -\frac{3}{2}, \frac{9}{4}, -\frac{27}{8}, \ldots$ has a sum to infinity. Justify your answer.

The common ratio is $-\frac{3}{2}$. Hence, as r does not satisfy $-1 < r < 1$, $\nexists$ S_∞.

Example 24

Find the sum to infinity of the geometric sequence $3, 2, \frac{4}{3}, \frac{8}{9}, \ldots$.

The first term is 3 and the common ratio is $\frac{2}{3}$. As $-1 < \frac{2}{3} < 1$, the sum to infinity exists. Hence,

$$S_\infty = \frac{3}{1 - 2/3}$$

$$S_\infty = \frac{9}{3 - 2}$$

$$S_\infty = 9$$

Example 25

Given that a geometric sequence has $S_\infty = 56$ and $a = 19$, find the common ratio.

$$\frac{19}{1 - r} = 56$$

$$1 - r = \frac{19}{56}$$

$$r = 1 - \frac{19}{56}$$

$$r = \frac{37}{56}$$

Example 26

Given that a geometric sequence has $S_\infty = \dfrac{3}{7}$ and $r = \dfrac{1}{7}$, find the first term.

$$\frac{a}{1 - 1/7} = \frac{3}{7}$$

$$\frac{7a}{6} = \frac{3}{7}$$

$$a = \frac{18}{49}$$

Expansion of 1/(1 – f(x))

There is an interesting link between infinite series and what may be viewed as an extension of the Binomial Theorem to the case $n = -1$.

Definition:

A **power series** is an expression of the form,

$$\sum_{i=0}^{\infty} a_i x^i = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots \quad (a_i \in \mathbb{R})$$

The aforementioned link is the content of the next theorem.

<u>Theorem:</u>

If $|x| < 1$, then,

$$(1 - x)^{-1} = \frac{1}{1 - x} = 1 + x + x^2 + x^3 + \ldots \stackrel{def}{=} \sum_{i=0}^{\infty} x^i$$

<u>Example 27</u>

Expand $(1 - 2x)^{-1}$, stating the range of values of x for which the expansion is valid.

$$(1 - 2x)^{-1} = \frac{1}{1 - (2x)} = 1 + 2x + (2x)^2 + (2x)^3 + \ldots$$

$$= 1 + 2x + 4x^2 + 8x^3 + \ldots$$

The expansion is valid for $|2x| < 1$, i.e. for $|x| < \frac{1}{2}$.

<u>Example 28</u>

Write $\frac{1}{1 + 3x}$ in the form $\sum_{i=0}^{\infty} (-1)^i k^i x^i$, stating the value of k.

$$\frac{1}{1 + 3x} = \frac{1}{1 - (-3x)} = \sum_{i=0}^{\infty} (-3x)^i$$

$$= \sum_{i=0}^{\infty} (-3)^i x^i$$

$$= \sum_{i=0}^{\infty} (-1)^i 3^i x^i$$

The value of k is 3.

Example 29

Write $\dfrac{1}{2 + 5x}$ in the form $p\sum_{i=0}^{\infty} (-1)^i k^i x^i$, stating the range of validity of the expansion as well as the values of p and k.

$$\frac{1}{2 + 5x} = \frac{1}{2}\left(\frac{1}{1 + (5/2)x}\right)$$

$$= \frac{1}{2}\left(\frac{1}{1 - (-5x/2)}\right)$$

$$= \frac{1}{2}\sum_{i=0}^{\infty} \left(-\frac{5x}{2}\right)^i$$

The expansion is valid for $\left|-\dfrac{5x}{2}\right| < 1$, i.e. for $|x| < \dfrac{2}{5}$. Continuing,

$$\frac{1}{2 + 5x} = \frac{1}{2}\sum_{i=0}^{\infty} (-1)^i \left(\frac{5}{2}\right)^i x^i$$

Thus, $p = \dfrac{1}{2}$ and $k = \dfrac{5}{2}$.

Example 30

Expand $\dfrac{2}{2 + 14 \sin 3x}$, stating the range of validity of the expansion, and write it in the form $\sum_{i=0}^{\infty} (-1)^i k^i (\sin 3x)^i$, stating the value of k.

$$\frac{2}{2 + 14 \sin 3x} = \frac{1}{1 + 7 \sin 3x}$$

$$= \frac{1}{1 - (-7 \sin 3x)}$$

$$= 1 + (-7 \sin 3x) + (-7 \sin 3x)^2 + (-7 \sin 3x)^3 + \dots$$

$$= 1 - 7 \sin 3x + 49 \sin^2 3x - 343 \sin^3 3x + \dots$$

which is valid for $|-7 \sin 3x| < 1$, i.e. for $|\sin 3x| < \frac{1}{7}$. In terms of the infinite sum,

$$\frac{2}{2 + 14 \sin 3x} = \sum_{i=0}^{\infty} (-1)^i 7^i (\sin 3x)^i$$

with $k = 7$.

<u>Definition:</u>

The **number e** is,

$$e \overset{def}{=} \lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n = 2 + \frac{1}{2\,!} + \frac{1}{3\,!} + \dots = \sum_{b=0}^{\infty} \frac{1}{b\,!}$$

<u>Theorem:</u>

The **exponential function** is,

$$e^x \overset{def}{=} \lim_{n \to \infty} \left(1 + \frac{x}{n}\right)^n$$

<u>Example 31</u>

State the exact value of $\lim_{n \to \infty} \left(1 + \frac{7}{n}\right)^n$ and write it to 8 significant figures.

The exact value is e^7. To 8 s.f., a calculator gives 1 096 · 633 2.

Finite Sums

Some special types of finite sums must be known.

Theorem:

The sum to n terms of the number 1 is,

$$\sum_{r=1}^{n} 1 = n$$

This result is supposed to be very obvious; adding up the number 1 n times gives the answer n.

Example 32

Find an expression for $\sum_{r=1}^{n} 3$.

$$\sum_{r=1}^{n} 3 = 3\sum_{r=1}^{n} 1$$

$$= 3n$$

The next result tells us what happens when we add up the sum of the first n natural numbers.

Theorem:

The sum of the first n natural numbers is,

$$\sum_{r=1}^{n} r = \frac{1}{2}n(n + 1)$$

Example 33

Find an expression for $\sum_{r=1}^{n} 8r$.

$$\sum_{r=1}^{n} 8r = 8 . \sum_{r=1}^{n} r$$

$$= 8 . \frac{1}{2} n(n + 1)$$

$$= 4n(n + 1)$$

The above 2 finite sums are often used in the following type of example.

Example 34

Express $\sum_{r=1}^{n} (7r - 5)$ in the form $Pn^2 + Qn$, stating the values of P and Q.

$$\sum_{r=1}^{n} (7r - 5) = \sum_{r=1}^{n} 7r - \sum_{r=1}^{n} 5$$

$$= 7\sum_{r=1}^{n} r - 5\sum_{r=1}^{n} 1$$

$$= \frac{7}{2} n(n + 1) - 5n$$

$$= \frac{7}{2} n^2 + \frac{7}{2} n - 5n$$

$$= \frac{7}{2} n^2 - \frac{3}{2} n$$

Hence, $P = \frac{7}{2}$ and $Q = -\frac{3}{2}$.

Other Finite Sums

Example 35

Express $-3 + 10 - 17 + 24 - \ldots - 59$ in the form $\sum_{r=1}^{n} (-1)^r (ar + b)$, stating the values of a, b and n.

The $(-1)^r$ serves to provide the alternating plus and minus signs. The $ar + b$ is indicative of an arithmetic sequence. Ignoring negatives, the differences are 7 and the first term is 3. Hence, the n^{th} term is given by $3 + (n - 1)7 = 7n - 4$. Counting up from 3 to 59 in 7's shows that $n = 9$. Thus, the required expression for the finite sum is,

$$\sum_{r=1}^{9} (-1)^r (7r - 4)$$

Therefore, $a = 7$, $b = -4$ and $n = 9$.