Systems of Equations

Prerequisites: Solving simultaneous equations in 2 variables; equation and graph of a straight line.

Maths Applications: Finding circle equations; finding matrix inverses; intersections of lines and planes.

Real-World Applications: Finding currents and voltages in electrical circuits; chemical reactions.

Linear Systems and the Augmented Matrix

Linear equations are very important objects of study in maths (and outside of it).

Definition:

A **linear equation in *n* variables** (aka **unknowns**) is an equation of the form,

$$a_1x_1 + a_2x_2 + \ldots + a_nx_n = b$$

where a_i , $b \in \mathbb{R}$.

The next definition extends the concept of 'simultaneous equations' to more than 2 equations in 2 unknowns.

Definition:

A **system of *m* linear equations in *n* unknowns** is a collection of linear equations,

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n &= b_m \end{aligned}$$

where a_{ij} , $b_i \in \mathbb{R}$.

Sometimes a system of m linear in n unknowns is referred to as an '$m \times n$ system'.

In this course, we will study the case of (at most) 3×3 systems.

The problem is to solve for the unknowns (if possible). There are 3 possibilities.

- Unique solution.
- Infinitely many solutions.
- No solution.

Definition:

A **consistent system** of linear equations is one that has a unique solution or infinitely many solutions.

Definition:

An **inconsistent system** of linear equations is one that has no solutions.

Example 1

The 2×2 system,

$$\begin{aligned} 2x + y &= 6 \\ x + y &= 2 \end{aligned}$$

has the unique solution $x = 4$, $y = -2$.

Example 2

The 2×2 system,

$$\begin{aligned} 2x + 2y &= 10 \\ x + y &= 5 \end{aligned}$$

has infinitely many solutions; for example, $(x, y) = (0, 5), (1, 4), (2, 3),$

$(1 \cdot 6, 3 \cdot 4)$, $(\pi, 1 - \pi)$, … or more generally, $(x, y) = (t, 1 - t)$ $(\forall\, t \in \mathbb{R})$.

<u>Example 3</u>

The 2×2 system,

$$\begin{aligned} 2x + y &= 3 \\ 2x + y &= 4 \end{aligned}$$

has no solutions, as equating the 2 equations leads to the absurd conclusion that $3 = 4$.

<u>Definition:</u>

An **underdetermined system** of linear equations is one with fewer equations than unknowns.

Note that underdetermined systems may be consistent or inconsistent (in the consistent case, there must be infinitely many solutions).

<u>Example 4</u>

The 1×2 system,

$$2x + y = 3$$

which is clearly underdetermined, has infinitely many solutions (similar to Example 2).

<u>Example 5</u>

The 2×3 system,

$$\begin{aligned} 2x + y + z &= 3 \\ 4x + 2y + 2z &= 7 \end{aligned}$$

which is obviously underdetermined, has no solutions (similar to Example 3).

Definition:

An **overdetermined system** of linear equations is one with more equations than unknowns.

Note that overdetermined systems may be consistent or inconsistent.

Example 6

The 3×2 system,

$$\begin{aligned} 2x &+ 4y = 14 \\ x &+ 2y = 7 \\ x &+ 2y = 3 \end{aligned}$$

is clearly overdetermined and inconsistent ($7 = 3$).

Example 7

The 3×2 system,

$$\begin{aligned} 2x &+ 4y = 14 \\ x &+ 2y = 7 \\ 3x &+ 6y = 21 \end{aligned}$$

is clearly overdetermined and has infinitely many solutions of the form $(x, y) = (7 - 2t, t)$ ($\forall\, t \in \mathbb{R}$).

Example 8

The 3×2 system,

$$\begin{aligned} 2x &+ 4y = 14 \\ x &+ 2y = 7 \\ x &+ 3y = 10 \end{aligned}$$

is clearly overdetermined and has a unique solution $x = 1$, $y = 3$.

With systems greater than 2 $\times$ 2, solving linear equations through traditional approaches such as substitution or elimination becomes very lengthy. However, a technique discovered in ancient China (about 200 B.C.E.) and independently in Europe by Isaac Newton can be used to solve linear systems in a much slicker manner.

The technique is named after Carl Friedrich Gauss who used the technique to solve a problem in astronomy involving a 6 $\times$ 6 system.

Gaussian Elimination

The technique involves picking out all the numbers in the system of equations and writing them in the same arrangement as given in the equations. Although we mainly study 3 $\times$ 3 systems, the following definitions apply in more general cases.

Definition:

The **Coefficient Matrix** of the system,

$$\begin{array}{ccccccc} ax & + & by & + & cz & = & j \\ dx & + & ey & + & fz & = & k \\ gx & + & hy & + & iz & = & l \end{array}$$

is the arrangement (aka **matrix**),

$$\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$$

whereas the **Augmented Matrix** is,

$$\left(\begin{array}{ccc|c} a & b & c & j \\ d & e & f & k \\ g & h & i & l \end{array}\right)$$

Gaussian Elimination works by considering the rows of the Augmented Matrix and applying some operations to the rows. These operations are called **Elementary Row Operations (EROs)**. Elementary Row Operations are of 3 types.

- Multiply a row by a non-zero scalar.
- Add or subtract 2 rows.
- Interchange 2 or more rows.

Each time an ERO is applied, the system changes. The EROs are designed to make the new system have the same solution (if it exists) of the original system. Quite often, the first 2 row operations are combined into the following operation.

- Add/subtract a multiple of one row to/from another row.

Applying an ERO is called **row reduction** or **row reducing** the Augmented Matrix. The Augmented Matrix is **row-reduced** into the following form.

Definition:

A matrix is in **echelon form** if (i) all non-zero rows are above any rows of zeros (ii) each non-zero row has more leading zeros than the previous row.

An example is,

$$\begin{pmatrix} a & b & c \\ 0 & e & f \\ 0 & 0 & i \end{pmatrix}$$

Gaussian Elimination involves transforming the Coefficient Matrix to echelon form. The existence and type of solution of the system essentially depends on the last row of the transformed Augmented Matrix.

Once the solution for one of the variables is obtained, the other solutions are obtained by **back-substitution** into the remaining equations.

No Solution

<u>Theorem:</u>

A system is inconsistent if the Augmented Matrix is transformed into,

$$\left(\begin{array}{ccc|c} a & b & c & j \\ 0 & e & f & k \\ 0 & 0 & 0 & l \end{array}\right) \qquad (l \neq 0)$$

<u>Example 9</u>

Show that the system of equations,

$$\begin{aligned} x + y + z &= 150 \\ x + 2y + 3z &= 100 \\ 2x + 3y + 4z &= 200 \end{aligned}$$

has no solution.

The Augmented Matrix is,

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 150 \\ 1 & 2 & 3 & 100 \\ 2 & 3 & 4 & 200 \end{array}\right)$$

$R_2 \to R_2 - R_1$
$R_3 \to R_3 - 2R_1$

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 150 \\ 0 & 1 & 2 & -50 \\ 0 & 1 & 2 & -100 \end{array}\right)$$

$R_3 \to R_3 - R_2$

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 150 \\ 0 & 1 & 2 & -50 \\ 0 & 0 & 0 & -50 \end{array}\right)$$

Hence, as the LHS is 0 but the RHS in non-zero, there is no solution.

Infinitely Many Solutions

Theorem:

A system has infinitely many solutions if the Augmented Matrix is transformed into,

$$\left(\begin{array}{ccc|c} a & b & c & j \\ 0 & e & f & k \\ 0 & 0 & 0 & 0 \end{array}\right)$$

Example 10

Find the value of k for which the following system has infinitely many solutions, and find the solutions.

$$\begin{aligned} x + 2y + 2z &= 11 \\ x - y + 3z &= 8 \\ 4x - y + kz &= 35 \end{aligned}$$

The Augmented Matrix is,

$$\left(\begin{array}{ccc|c} 1 & 2 & 2 & 11 \\ 1 & -1 & 3 & 8 \\ 4 & -1 & k & 35 \end{array}\right)$$

$R_2 \to R_2 - R_1$
$R_3 \to R_3 - 4R_1$

$$\left(\begin{array}{ccc|c} 1 & 2 & 2 & 11 \\ 0 & -3 & 1 & -3 \\ 0 & -9 & k-8 & -9 \end{array}\right)$$

$R_3 \to R_3 - 3R_2$

$$\left(\begin{array}{ccc|c} 1 & 2 & 2 & 11 \\ 0 & -3 & 1 & -3 \\ 0 & 0 & k-11 & 0 \end{array}\right)$$

For there to be infinitely many solutions, the last row must consist entirely of zeros; hence, $k - 11 = 0 \Rightarrow k = 11$. The remaining 2 equations can be translated into x, y and z form as,

$$\begin{aligned} x + 2y + 2z &= 11 \\ -3y + z &= -3 \end{aligned}$$

Putting $z = t$ into the second equation gives $y = \dfrac{t + 3}{3}$ and the first equation then gives, $x = 11 - 2t - \dfrac{2(t + 3)}{3} \Rightarrow x = \dfrac{27 - 8t}{3}$.
Hence, the solutions are,

$$x = \frac{27 - 8t}{3},\ y = \frac{t + 3}{3},\ z = t \quad (\forall\, t \in \mathbb{R})$$

Unique Solution

<u>Theorem:</u>

A system has a unique solution if the Augmented Matrix is transformed into,

$$\left(\begin{array}{ccc|c} a & b & c & j \\ 0 & e & f & k \\ 0 & 0 & i & l \end{array}\right) \qquad (i \neq 0)$$

<u>Example 11</u>

Find the solution to the system,

$$\begin{aligned} x + y + z &= 2 \\ 4x + 2y + z &= 4 \\ x - y + z &= 4 \end{aligned}$$

The Augmented Matrix is,

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 4 & 2 & 1 & 4 \\ 1 & -1 & 1 & 4 \end{array}\right)$$

$R_2 \to R_2 - 4R_1$
$R_3 \to R_3 - R_1$

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & -2 & -3 & -4 \\ 0 & -2 & 0 & 2 \end{array}\right)$$

$R_2 \leftrightarrow R_3$

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & -2 & 0 & 2 \\ 0 & -2 & -3 & -4 \end{array}\right)$$

$R_3 \to R_3 - R_2$

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & -2 & 0 & 2 \\ 0 & 0 & -3 & -6 \end{array}\right)$$

In terms of the unknowns, this is,

$$\begin{aligned} x + y + z &= 2 \\ -2y &= 2 \\ -3z &= -6 \end{aligned}$$

The third and second equations respectively give $z = 2$ and $y = -1$. Back-substitution into the first equation then gives $x = 2 - 2 + 1 = 1$. Hence, the unique solution is $x = 1$, $y = -1$, $z = 2$.

Ill-Conditioned Systems

Definition:

A system is **ill-conditioned** when changing the entries of the Augmented Matrix by a small amount induces a large change in the solution of the system.

In this course, ill-conditioning will be studied only for 2×2 systems.

Example 12

The system,

$$\begin{aligned} 2x + y &= 4 \\ 2x + 1 \cdot 01y &= 4 \cdot 02 \end{aligned}$$

has solution $x = 1$, $y = 2$. Changing the coefficients slightly thus,

$$\begin{aligned} 2x + y &= 3 \cdot 8 \\ 2 \cdot 02x + y &= 4 \cdot 02 \end{aligned}$$

yields a system which has solution $x = 11$, $y = -18 \cdot 2$. So, a small change in the coefficients leads to a large change in the solution. This system is ill-conditioned.

Notice that in either system, the equations represent lines whose gradients are almost equal (the lines are almost parallel). In fact, this is the best way to think about ill-conditioned systems. 2 lines that are almost parallel will intersect at a given point; if the gradients of the lines are changed by a small amount, then the intersection point can be very different.

Example 13

The system,

$$\begin{aligned} 2x + 9y &= 17 \\ 3x - 5y &= 4 \end{aligned}$$

is not ill-conditioned, as the gradients of the lines are not very similar (one is negative and the other is positive).