Vectors, Lines and Planes

Prerequisites: Adding, subtracting and scalar multiplying vectors; calculating angles between vectors.

Maths Applications: Describing geometric transformations.

Real-World Applications: Description of forces; solar sailing; rotational mechanics; planetary orbits.

Direction Ratios and Direction Cosines

This short section contains some simple ideas that are useful in later sections.

Definition:

The **direction ratio** of a vector is the ratio of its components in order from first to last.

Thus, the direction ratio of $\mathbf{p} = \begin{pmatrix} p_1 \\ p_2 \end{pmatrix}$ is $p_1 : p_2$ and the direction ratio of $\mathbf{q} = \begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix}$ is $q_1 : q_2 : q_3$.

Theorem:

Vectors have equal direction ratios if and only if they are parallel.

Example 1

Determine which of the vectors defined by $\mathbf{p} = \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}$, $\mathbf{q} = \begin{pmatrix} -2 \\ 1 \\ -3 \end{pmatrix}$ and $\mathbf{r} = \begin{pmatrix} 4 \\ -2 \\ 6 \end{pmatrix}$ are parallel.

Working out the direction ratios of each vector (and simplifying), we get,

$$p_1 : p_2 : p_3 = 2 : -1 : 3$$

$$q_1 : q_2 : q_3 = 2 : -1 : 3$$

$$r_1 : r_2 : r_3 = 2 : -1 : 3$$

Hence, as the direction ratios of all 3 vectors are equal, all 3 vectors are parallel to each other.

<u>Definition:</u>

If α, β and γ are the angles the vector **p** makes with the x, y and z axes respectively, and **u** is a unit vector in the direction of **p**, the **direction cosines** of **p** are cos α, cos β and cos γ and satisfy,

$$\mathbf{u} = \begin{pmatrix} \cos\alpha \\ \cos\beta \\ \cos\gamma \end{pmatrix} \Rightarrow \cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$$

In other words, the direction cosines of a vector **p** are the components of a unit vector in the direction of **p**.

<u>Example 2</u>

Find the direction cosines of the vector **p** = 6 **i** + 18 **j** - 22 **k**.

The magnitude of **p** is $|\mathbf{p}| = \sqrt{6^2 + 18^2 + (-22)^2} = \sqrt{844}$. Hence, a unit vector in the direction of **p** is,

$$\mathbf{u} = \begin{pmatrix} \frac{6}{\sqrt{844}} \\ \frac{18}{\sqrt{844}} \\ -\frac{22}{\sqrt{844}} \end{pmatrix}$$

and thus the direction cosines of **p** are $\cos\alpha = \frac{6}{\sqrt{844}}$, $\cos\beta = \frac{18}{\sqrt{844}}$ and $\cos\gamma = -\frac{22}{\sqrt{844}}$.

The Vector Product

Vector Form

Definition:

The **vector product** (aka **cross product**) of 2 vectors **a** and **b**, where θ is the angle from **a** to **b**, and **n** is a unit vector at right angles to both **a** and **b**, is defined as,

$$\mathbf{a} \times \mathbf{b} \overset{def}{=} |\mathbf{a}|\ |\mathbf{b}| \sin\theta\ \mathbf{n}$$

Imagine grasping a pole with your right hand. Then the direction in which your fingers curl around the pole indicate the direction of the angle from **a** to **b** and the thumb points in the direction of **n**.

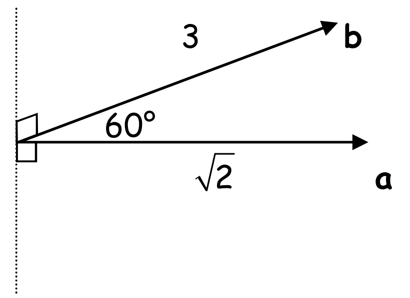
Note that while the scalar product can be defined in any number of dimensions, the vector product is a 3D-only quantity. Also, the order in which the vector product is taken matters (as opposed to the scalar product) – figure out why.

Also note that the vector product is a vector and is at right angles to both **a** and **b**.

The vector product arises in many areas of physics, especially rotational mechanics.

Example 3

Calculate the vector product **a** × **b**, as well as its magnitude, in the following diagram.



Using the definition of the vector product,

$$\mathbf{a} \times \mathbf{b} = |\mathbf{a}|\ |\mathbf{b}| \sin\theta\ \mathbf{n}$$

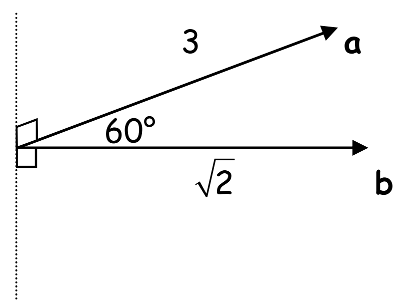
where **n** points upwards, gives,

$$\mathbf{a} \times \mathbf{b} = 3\sqrt{2}\ \sin 60°\ \mathbf{n}$$

$$= \frac{3\sqrt{3}}{\sqrt{2}}\ \mathbf{n}$$

<u>Example 4</u>

Calculate the vector product **a** × **b**, as well as its magnitude, in the following diagram.



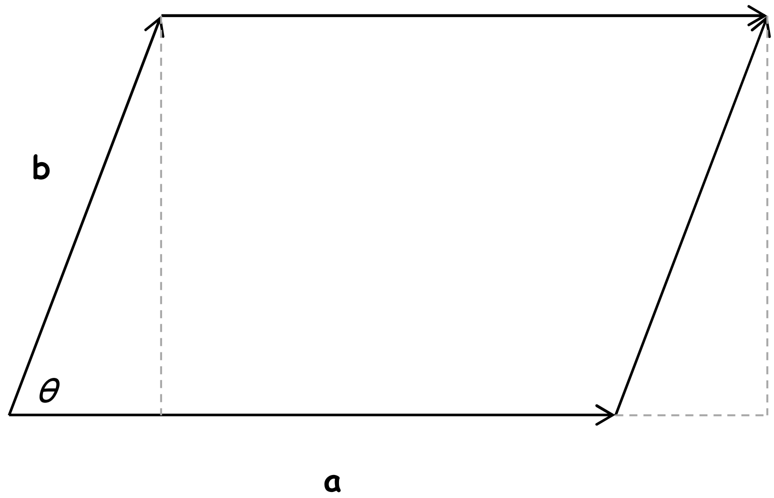
We have,

$$\mathbf{a} \times \mathbf{b} = -\frac{3\sqrt{3}}{\sqrt{2}}\ \mathbf{n}$$

where **n** points upwards.

Geometrical Interpretation of the Vector Product Magnitude

The magnitude of the vector product has a geometric interpretation in terms of area. By considering the following diagram,



we see that the following theorem is true.

<u>Theorem:</u>

The area of a parallelogram with side lengths $|\mathbf{a}|$ and $|\mathbf{b}|$ is $|\mathbf{a} \times \mathbf{b}|$.

<u>Theorem:</u>

The 3 unit vectors, **i**, **j** and **k** satisfy,

$$\mathbf{i} \times \mathbf{j} = \mathbf{k}$$

$$\mathbf{j} \times \mathbf{k} = \mathbf{i}$$

$$\mathbf{k} \times \mathbf{i} = \mathbf{j}$$

and

$$\mathbf{i} \times \mathbf{i} = \mathbf{j} \times \mathbf{j} = \mathbf{k} \times \mathbf{k} = \mathbf{0}$$

The Scalar Triple Product

<u>Definition:</u>

The **scalar triple product** of 3 vectors **a**, **b** and **c** is,

$$[\mathbf{a}, \mathbf{b}, \mathbf{c}] \equiv \mathbf{a} \bullet (\mathbf{b} \times \mathbf{c}) \overset{def}{=} \left| \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix} \right|$$

<u>Example 5</u>

Work out the scalar triple product [**b**, **a**, **c**] for the vectors
a = 3 **i** + 2 **j** - **k**, **b** = - 2 **i** + 5 **k** and **c** = - **i** + **j** + 4 **k**.

We have,

$$[\mathbf{b}, \mathbf{a}, \mathbf{c}] = \mathbf{b} \bullet (\mathbf{a} \times \mathbf{c})$$

$$= \left| \begin{pmatrix} b_1 & b_2 & b_3 \\ a_1 & a_2 & a_3 \\ c_1 & c_2 & c_3 \end{pmatrix} \right|$$

$$= \left| \begin{pmatrix} -2 & 0 & 5 \\ 3 & 2 & -1 \\ -1 & 1 & 4 \end{pmatrix} \right|$$

$$= -18 - 0 + 25$$

$$= 7$$

The scalar triple product has a geometrical interpretation as the volume of a parallelepiped (roughly speaking, a 3D parallelogram). Make a sketch !

Theorem:

The volume of a parallelepiped with side lengths $|\mathbf{a}|$, $|\mathbf{b}|$ and $|\mathbf{c}|$ is given by any of the following 6 expressions,

$$\mathbf{a} \bullet (\mathbf{b} \times \mathbf{c}) = \mathbf{a} \times (\mathbf{b} \bullet \mathbf{c})$$

$$\mathbf{b} \bullet (\mathbf{c} \times \mathbf{a}) = \mathbf{b} \times (\mathbf{c} \bullet \mathbf{a})$$

$$\mathbf{c} \bullet (\mathbf{a} \times \mathbf{b}) = \mathbf{c} \times (\mathbf{a} \bullet \mathbf{b})$$

This theorem is useful in proving the last 2 properties in the next subsection.

Properties of the Vector Product

- $\mathbf{a} \times \mathbf{a} = \mathbf{0}$
- $\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}$
- $\mathbf{a} \times (\mathbf{b} + \mathbf{c}) = (\mathbf{a} \times \mathbf{b}) + (\mathbf{a} \times \mathbf{c})$
- $(\mathbf{a} + \mathbf{b}) \times \mathbf{c} = (\mathbf{a} \times \mathbf{c}) + (\mathbf{b} \times \mathbf{c})$

Example 6

For $\mathbf{a} = \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}$, $\mathbf{a} \times \mathbf{a} = |\mathbf{a}|\ |\mathbf{a}| \sin 0° \ \mathbf{n} = \mathbf{0}$.

Notice that the 'zero' is the zero vector $\mathbf{0}$ and not the zero scalar 0.

Example 7

Show that $(\mathbf{a} - \mathbf{b}) \times (\mathbf{a} + \mathbf{b}) = 2(\mathbf{a} \times \mathbf{b})$.

$$(\mathbf{a} - \mathbf{b}) \times (\mathbf{a} + \mathbf{b}) = (\mathbf{a} \times \mathbf{a}) + (\mathbf{a} \times \mathbf{b}) - (\mathbf{b} \times \mathbf{a}) - (\mathbf{b} \times \mathbf{b})$$

$$= \mathbf{0} + (\mathbf{a} \times \mathbf{b}) + (\mathbf{a} \times \mathbf{b}) - \mathbf{0}$$

$$= 2(\mathbf{a} \times \mathbf{b})$$

Theorem:

If the vector product of 2 vectors is **0**, then they are parallel.

Example 8

Show that any vector is parallel to itself.

As $\mathbf{a} \times \mathbf{a} = \mathbf{0}$, **a** is parallel to itself.

Theorem:

If 2 non-zero vectors are parallel, then their vector product is **0**.

Example 9

Show that $\mathbf{a} \times (2\mathbf{a} + \mathbf{b}) + (\mathbf{b} \times \mathbf{a}) = \mathbf{0}$.

$$\begin{aligned}\mathbf{a} \times (2\mathbf{a} + \mathbf{b}) + (\mathbf{b} \times \mathbf{a}) &= 2\mathbf{a} \times \mathbf{a} + (\mathbf{a} \times \mathbf{b}) + (\mathbf{b} \times \mathbf{a}) \\ &= \mathbf{0} + (\mathbf{a} \times \mathbf{b}) - (\mathbf{a} \times \mathbf{b}) \\ &= \mathbf{0}\end{aligned}$$

Component Form

Theorem:

Given 2 vectors $\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$, the component form of the vector product is,

$$\mathbf{a} \times \mathbf{b} = (a_2 b_3 - a_3 b_2)\,\mathbf{i} + (a_3 b_1 - a_1 b_3)\,\mathbf{j} + (a_1 b_2 - a_2 b_1)\,\mathbf{k}$$

Example 10

Calculate the vector product of **a** = 2 **i** - 5 **j** + 3 **k** and **b** = 4 **i** - 11 **j** - 7 **k**.

$$\mathbf{a} \times \mathbf{b} = (35 + 33)\,\mathbf{i} + (12 + 14)\,\mathbf{j} + (-22 + 20)\,\mathbf{k}$$

$$= 68\,\mathbf{i} + 26\,\mathbf{j} - 2\,\mathbf{k}$$

Example 11

Find a unit vector in the direction of **s** x **t**, where **s** = **i** + 2 **j** + 3 **k** and **t** = - 9 **i** + 3 **k**.

$$\mathbf{s} \times \mathbf{t} = 6\,\mathbf{i} - 30\,\mathbf{j} + 18\,\mathbf{k}$$

The magnitude of **s** x **t** is $6\sqrt{35}$. Hence, a unit vector in the direction of **s** x **t** is

$$\begin{pmatrix} \frac{1}{\sqrt{35}} \\ -\frac{5}{\sqrt{35}} \\ \frac{3}{\sqrt{35}} \end{pmatrix}$$

Example 12

Calculate the exact value of the sine of the angle between the vectors **s** = **i** + **j** + 2 **k** and **t** = - 5 **j** + **k**.

$$\mathbf{s} \times \mathbf{t} = |\mathbf{s}|\;|\mathbf{t}|\;\sin\theta\;\mathbf{n}$$

We have **s** x **t** = 11 **i** - **j** - 5 **k**, $|\mathbf{s}| = \sqrt{6}$ and $|\mathbf{t}| = \sqrt{26}$. A unit vector in the direction of **s** x **t** is $\mathbf{n} = \begin{pmatrix} \frac{11}{7\sqrt{3}} \\ -\frac{1}{7\sqrt{3}} \\ -\frac{5}{7\sqrt{3}} \end{pmatrix}$. Hence,

$$\begin{pmatrix} 11 \\ -1 \\ -5 \end{pmatrix} = \sqrt{6}\ \sqrt{26}\ \sin\theta \begin{pmatrix} \frac{11}{7\sqrt{3}} \\ -\frac{1}{7\sqrt{3}} \\ -\frac{5}{7\sqrt{3}} \end{pmatrix}$$

Taking the scalar product of each side of the above equation with **n** gives,

$$\frac{149}{7\sqrt{3}} = \sqrt{6}\ \sqrt{26}\ \sin\theta$$

$$7\sqrt{3} = \sqrt{6}\ \sqrt{26}\ \sin\theta$$

$$\sin\theta = \frac{7}{2\sqrt{13}}$$

Example 13

Show that no value of p makes the vectors $\mathbf{a} = 2p\,\mathbf{i} + \mathbf{j} - 3\,\mathbf{k}$ and $\mathbf{b} = -7\,\mathbf{i} + p\,\mathbf{j} - \mathbf{k}$ parallel.

For the vectors to be parallel, we require $\mathbf{a} \times \mathbf{b} = \mathbf{0}$. We have,

$$\mathbf{a} \times \mathbf{b} = \begin{pmatrix} 3p - 1 \\ 2p + 21 \\ 2p^2 + 7 \end{pmatrix}$$

The third component obviously cannot equal 0. Hence, this vector clearly cannot equal **0**. Thus the given vectors cannot possibly be parallel.

Equations of Planes

The equation of a plane can be found in 3 ways, depending on the information given.

- 1 point in the plane and a vector at right angles to the plane.
- 2 vectors in the plane.
- 3 points in the plane.

Definition:

A vector is **parallel to a plane** if it lies in the plane.

Definition:

A **normal vector to a plane** is one that is at right angles to any vector in the plane.

Definition:

The **Cartesian equation of a plane** containing a point P (x, y, z) is,

$$ax + by + cz = d \qquad (a, b, c, d, x, y, z \in \mathbb{R})$$

The equation of a plane is not unique. Multiplying the equation of a plane by a non-zero scalar will give an apparently different equation, but it's really the same (as the scalar can be divided out).

Example 14

Determine whether or not the planes $2x - 5y + 3z = 4$ and $6x - 15y + 9z = 12$ are the same.

As the second plane equation is just 3 times the first plane equation, the 2 planes are identical.

Example 15

Determine whether the points P (2, 0, 0) and Q (1, 1, 1) lie on the plane $2x - 5y + 3z = 4$.

As $2(2) - 5(0) + 3(0) = 4$, P lies on the plane.

As $2(1) - 5(1) + 3(1) = 0 \neq 4$, Q doesn't lie on the given plane.

Theorem:

A normal vector to the plane $ax + by + cz = d$ is,

$$\mathbf{n} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

Note that a plane has 2 possible directions for a normal. The other normal for the above plane would be $\begin{pmatrix} -a \\ -b \\ -c \end{pmatrix}$.

Theorem:

Parallel planes have the same direction ratios for their normals.

Cartesian Equation of a Plane Given Normal and 1 Point on Plane

Given a normal and 1 point on the plane, the Cartesian equation of the plane can be quickly found by determining d.

Example 16

Determine the Cartesian equation of the plane passing through the point (2, 3, 1) and which has normal vector $\begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix}$.

Substituting the given point and normal components into the generic Cartesian equation gives $d = 2(2) + 3(0) + 1(-1) = 3$. Hence, the Cartesian equation is,

$$2x - z = 3$$

Cartesian Equation of a Plane Given 2 Vectors and a Point in the Plane

By taking the vector product of 2 vectors lying in the plane, a normal can be determined. Together with the given point that lies on the plane, we are back to the previous case.

Example 17

Find the equation of the plane passing through (3, 1, 1) and containing the vectors $\begin{pmatrix} 2 \\ 5 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}$.

A normal to the plane is,

$$\begin{pmatrix} 2 \\ 5 \\ 1 \end{pmatrix} \times \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} -6 \\ 4 \\ -8 \end{pmatrix}$$

Then $d = 3(-6) + 1(4) + 1(-8) = -22$. Hence, the Cartesian equation is, upon simplification,

$$3x - 2y + 4z = 11$$

Cartesian Equation of a Plane Given 3 Points in the Plane

Given 3 points on a plane, 2 vectors emanating from one point can be formed and thus the equation can be determined from the previous case.

Example 18

Determine the equation of the plane containing the points A (1, −6, 0), B (−4, 2, −5) and C (−2, 4, 1).

The vectors $\overrightarrow{AB}$ and $\overrightarrow{AC}$ are,

$$\overrightarrow{AB} = \begin{pmatrix} -5 \\ 8 \\ -5 \end{pmatrix} \text{ and } \overrightarrow{AC} = \begin{pmatrix} -3 \\ 10 \\ 1 \end{pmatrix}$$

Taking the vector product gives us a normal to the plane,

$$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{pmatrix} 58 \\ 20 \\ -26 \end{pmatrix}$$

Then $d = 58(1) + 20(-6) - 26(0) = -62$. Hence, an equation for the plane is,

$$58x + 20y - 26z = -62$$

which becomes, upon simplification,

$$29x + 10y - 13z = -31$$

Notice that the vector product can be taken in any order; the negative sign will cancel out from the plane equation.

<u>Definition:</u>

For a point A (with position vector **a**) in a plane, and 2 vectors **b** and **c** parallel to the plane, a **vector equation** (aka **parametric equation**) **of a plane** for a point R (with position vector **r**), where t and u are real parameters, is,

$$\mathbf{r} = \mathbf{a} + t\,\mathbf{b} + u\,\mathbf{c}$$

Note that the vectors **b** and **c** are parallel to (in) the plane. If we are told 3 points that lie in the plane, then **b** and **c** are formed by taking differences of the position vectors of these 3 points.

<u>Example 19</u>

Determine whether the points F $(1, -1, 0)$ and G $(-8, 9, 14)$ lie on
$\mathbf{r} = (1 - 3t + 3u)\,\mathbf{i} + (2 + t + 3u)\,\mathbf{j} + (-1 + 3t + 3u)\,\mathbf{k}$.

If F lies on the plane then,

$$1 - 3t + 3u = 1$$

$$2 + t + 3u = -1$$

$$-1 + 3t + 3u = 0$$

Adding the first and third equations gives $u = 1/6$. Substituting this into either the first or third equation then gives $t = 1/6$. However, these solutions do not satisfy the second equation (check !). Thus, F does not lie on the plane.

A similar analysis for G shows that $u = 1$ and $t = 4$ satisfy,

$$1 - 3t + 3u = -8$$

$$2 + t + 3u = 9$$

$$-1 + 3t + 3u = 14$$

Hence, G lies on the plane.

Vector Equation of a Plane Given Normal and 1 Point on Plane

Example 20

Determine a vector equation of the plane passing through the point D (2, 3, 4) and which has normal vector $2\,\mathbf{i} + 4\,\mathbf{k}$.

To find a vector equation, 2 vectors lying in the plane must be found. This is done by finding 2 vectors at right angles to the normal. Thus, 2 solutions must be found to the equation,

$$\begin{pmatrix} 2 \\ 0 \\ 4 \end{pmatrix} \bullet \begin{pmatrix} p \\ q \\ r \end{pmatrix} = 0$$

By inspection, 2 solutions are $\begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 3 \\ 0 \end{pmatrix}$. There are infinitely many solutions (draw a picture !).

Hence, a vector equation for the plane is,

$$\mathbf{r} = \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix} + t\begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix} + u\begin{pmatrix} 0 \\ 3 \\ 0 \end{pmatrix}$$

which can be written as,

$$\mathbf{r} = (2 - 2t)\,\mathbf{i} + (3 + 3u)\,\mathbf{j} + (4 + t)\,\mathbf{k}$$

Vector Equation of a Plane Given 2 Vectors in the Plane

<u>Example 21</u>

Find a vector equation of the plane passing through W (1, 1, 1) and parallel to $\mathbf{i} + 6\,\mathbf{j}$ and $5\,\mathbf{i} + 3\,\mathbf{j} - \mathbf{k}$.

A vector equation is,

$$\mathbf{r} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + t\begin{pmatrix} 1 \\ 6 \\ 0 \end{pmatrix} + u\begin{pmatrix} 5 \\ 3 \\ -1 \end{pmatrix}$$

$$\mathbf{r} = (1 + t + 5u)\,\mathbf{i} + (1 + 6t + 3u)\,\mathbf{j} + (1 - u)\,\mathbf{k}$$

Vector Equation of a Plane Given 3 Points in the Plane

<u>Example 22</u>

Determine the equation of the plane containing the points A (1, −6, 0), B (−4, 2, −5) and C (−2, 4, 1).

Two vectors lying in the plane are,

$$\overrightarrow{AB} = \begin{pmatrix} -5 \\ 8 \\ -5 \end{pmatrix} \text{ and } \overrightarrow{AC} = \begin{pmatrix} -3 \\ 10 \\ 1 \end{pmatrix}$$

Hence, a vector equation for the plane is,

$$\mathbf{r} = \begin{pmatrix} 1 \\ -6 \\ 0 \end{pmatrix} + t\begin{pmatrix} -5 \\ 8 \\ -5 \end{pmatrix} + u\begin{pmatrix} -3 \\ 10 \\ 1 \end{pmatrix}$$

$$\mathbf{r} = (1 - 5t - 3u)\,\mathbf{i} + (-6 + 8t + 10u)\,\mathbf{j} + (-5t + u)\,\mathbf{k}$$

Converting Between the Cartesian and Vector Forms

<u>Example 23</u>

Find the Cartesian equation of the plane which has vector equation
$\mathbf{r} = (3 - t + 2u)\,\mathbf{i} + (1 + 7t - 4u)\,\mathbf{j} + (1 - 5t + u)\,\mathbf{k}$.

Rewriting the vector **r** gives,

$$\mathbf{r} = \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix} + t\begin{pmatrix} -1 \\ 7 \\ -5 \end{pmatrix} + u\begin{pmatrix} 2 \\ -4 \\ 1 \end{pmatrix}$$

which shows that the point (3, 1, 1) lies on the plane and that the vectors $\begin{pmatrix} -1 \\ 7 \\ -5 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ -4 \\ 1 \end{pmatrix}$ lie in the plane. To find the Cartesian equation, we need a point in the plane (which we have) and a normal that can be constructed in the usual way,

$$\mathbf{n} = \begin{pmatrix} -1 \\ 7 \\ -5 \end{pmatrix} \times \begin{pmatrix} 2 \\ -4 \\ 1 \end{pmatrix} = \begin{pmatrix} -13 \\ -9 \\ -10 \end{pmatrix}$$

Thus, the Cartesian equation is,

$$-\ 13x\ -\ 9y\ -\ 10z\ =\ \begin{pmatrix}-13\\-9\\-10\end{pmatrix} \bullet \begin{pmatrix}3\\1\\1\end{pmatrix}\ =\ -\ 58$$

$$13x\ +\ 9y\ +\ 10z\ =\ 58$$

Example 24

Find a vector equation for the plane which has Cartesian equation $5x\ -\ 2y\ +\ 4z\ =\ 13$.

A normal to the plane is $\begin{pmatrix}5\\-2\\4\end{pmatrix}$. Two vectors at right angles to this normal are $\begin{pmatrix}2\\3\\-1\end{pmatrix}$ and $\begin{pmatrix}0\\2\\1\end{pmatrix}$. Again, there are infinitely many choices for these 2 vectors. By inspection, the point $(1,\ -2,\ 1)$ lies on the plane. A vector equation for the plane is thus,

$$\mathbf{r}\ =\ \begin{pmatrix}1\\-2\\1\end{pmatrix}\ +\ t\begin{pmatrix}2\\3\\-1\end{pmatrix}\ +\ u\begin{pmatrix}0\\2\\1\end{pmatrix}$$

$$\mathbf{r}\ =\ (1\ +\ 2t)\,\mathbf{i}\ +\ (-2\ +\ 3t\ +\ 2u)\,\mathbf{j}\ +\ (1\ -\ t\ +\ u)\,\mathbf{k}$$

Equations of Lines

Equations of lines can be determined in 2 ways, depending on the information given.

- 1 point on the line and a direction for the line.
- 2 points on the line.

Definition:

A **vector equation for a line** in 3D with **direction vector u** = $a\,\mathbf{i} + b\,\mathbf{j} + c\,\mathbf{k}$ passing through a point A(x_1, y_1, z_1) (with position vector **a**) and a general point P(x, y, z) with position vector **p** is,

$$\mathbf{p} = \mathbf{a} + t\,\mathbf{u} \qquad (t \in \mathbb{R})$$

Definition:

The **parametric form of a line**, using the notation in the previous definition, is,

$$x = x_1 + at, \quad y = y_1 + bt, \quad z = z_1 + ct$$

Definition:

The **symmetric form of a line** (aka **standard form** or **canonical form**), using the notation in the previous definition (assuming none of a, b or c equal 0), is,

$$\frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{c} = t$$

When at least one of a, b or c equal 0, we abandon the symmetric form and use the parametric form instead.

It is easy to convert between the vector, parametric and symmetric forms of a line.

As with planes, line equations are not unique.

Example 25

Show that the lines with equations $\dfrac{x - 1}{2} = \dfrac{y + 2}{4} = \dfrac{z + 4}{6}$ and $\dfrac{x - 2}{1} = \dfrac{y}{2} = \dfrac{z + 1}{3}$ are the same.

Lines are the same if their direction vectors are parallel and both lines pass through the same point. The first line has direction $\begin{pmatrix} 2 \\ 4 \\ 6 \end{pmatrix}$ and the second line has direction $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$. Combining the first line equation (with parameter s) and the second line equation (with parameter t) gives one equation (repeated thrice), namely, $2s = 1 + t$. This equation has infinitely many solutions. In other words, infinitely many common points lie on both lines. Hence, as the directions are parallel, the lines are the same.

Example 26

Determine whether or not the points B $(-2, -7, -3)$ and C $(1, 4, 1)$ lie on the line $\frac{x - 1}{3} = \frac{y + 2}{5} = \frac{z - 8}{11}$.

The parametric equations for this line are,

$$x = 1 + 3t, \quad y = -2 + 5t, \quad z = 8 + 11t$$

Substituting the coordinates of B into the parametric equations gives $t = -1$ for each equation. Hence, as there is a unique t value, B lies on the line.

Doing the same for C shows that $t = 0$, $t = 6/5$ and $t = -7/11$. As there is not a unique t value, C does not lie on the line.

Equation of a Line Given 1 Point on the Line and a Direction

Example 27

Find the vector, parametric and symmetric forms of the line passing through T (4, 11, 17) and which is parallel to $3\,\mathbf{i} + 5\,\mathbf{j} - 8\,\mathbf{k}$.

The vector equation is,

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 4 \\ 11 \\ 17 \end{pmatrix} + t\begin{pmatrix} 3 \\ 5 \\ -8 \end{pmatrix}$$

The parametric equations are,

$$x = 4 + 3t, \quad y = 11 + 5t, \quad z = 17 - 8t$$

The symmetric form is,

$$\frac{x - 4}{3} = \frac{y - 11}{5} = \frac{z - 17}{-8} = t \quad (t \in \mathbb{R})$$

Example 28

Equation of a Line Given 2 Points on the Line

Find the vector form of the line that passes through A (1, 4, 6) and B (3, −3, −3).

A direction vector for the line is $\begin{pmatrix} 2 \\ -7 \\ -9 \end{pmatrix}$. The vector form of the line is thus,

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 4 \\ 6 \end{pmatrix} + t\begin{pmatrix} 2 \\ -7 \\ -9 \end{pmatrix}$$

Of course, this is just one of the vector forms.

Example 29

Find the value of k for which the lines $\frac{x + 27}{6} = \frac{y - 4}{5} = \frac{z}{-3}$ and $\frac{x - 12}{1} = \frac{y - 39}{3} = \frac{z - 2}{k}$ are perpendicular.

For the lines to be perpendicular, the direction vectors of the lines must have zero scalar product. Hence,

$$6 + 15 - 3k = 0$$

$$3k = 21$$

$$k = 7$$

Intersections of Lines

Angle Between 2 Lines

The angle between 2 lines is the acute angle made by their direction vectors.

Example 30

Find the angle made by the lines $\frac{x + 27}{2} = \frac{y - 4}{1} = \frac{z}{-3}$ and $\frac{x - 12}{1} = \frac{y - 39}{3} = \frac{z - 2}{6}$.

The directions of the lines are $\begin{pmatrix} 2 \\ 1 \\ -3 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 3 \\ 6 \end{pmatrix}$. The scalar product equations give,

$$\cos\theta = \frac{-13}{\sqrt{14}\sqrt{46}}$$

$$\cos\theta = -0 \cdot 512\ldots$$

$$\theta = 120 \cdot 8^\circ$$

The acute angle between the lines is thus $59 \cdot 2^\circ$.

Definition:
Lines in 3D are **skew** if they neither intersect nor are parallel.

In 2D, 2 lines can intersect in 1 point, intersect in infinitely many points (are coincident) or not intersect (are parallel).

In 3D, the same situation occurs, except that in case of no intersection, the lines can be either parallel or skew.

Intersection of 2 Lines in a Point

Example 31

Find the point of intersection of the lines $x - 5 = -(y + 2) = z$ and $\frac{x - 12}{5} = \frac{y + 3}{-2} = \frac{z - 5}{4}$.

Parametrically, the lines take the respective forms,

$$x = 5 + t, \quad y = -2 - t, \quad z = t$$

$$x = 5s + 12, \quad y = -3 - 2s, \quad z = 5 + 4s$$

Equating components, we find that,

$$t = 5s + 7$$

$$t = 2s + 1$$

$$t = 4s + 5$$

These are easily solved to give $s = -2$ and $t = -3$. The unique solution for s and t means that the lines intersect at one point only. The intersection point is thus $(2, 1, -3)$.

Intersection of 2 Lines in a Line

Example 32

Show that the lines $\frac{x - 12}{5} = \frac{y + 3}{-2} = \frac{z - 5}{4}$ and $\frac{x - 12}{10} = \frac{y + 3}{-4} = \frac{z - 5}{8}$ intersect in infinitely many points.

The parametric equations, when equated, give $t = 2s$ (repeated thrice). There are infinitely many solutions to this equation. Hence, the lines intersect in infinitely many points.

The quick way to get the answer is to note that both lines clearly pass through the point (12, −3, 5) and have direction vectors that are parallel. Hence, the lines are one and the same (there is only 1 line !).

Non-Intersection of 2 Parallel Lines

Example 33

Show that the lines $\frac{x - 12}{5} = \frac{y + 3}{-2} = \frac{z - 5}{4}$ and $\frac{x - 3}{10} = \frac{y + 7}{-4} = \frac{z - 1}{8}$ are parallel.

Note that the lines clearly have direction vectors that are parallel. It still needs to be checked whether or not the lines meet. The equated parametric equations boil down to,

$$5t - 10s = -9$$

$$t - 2s = 2$$

$$t - 2s = -1$$

The last 2 equations are obviously inconsistent, so the lines do not intersect; together with the fact that the lines have parallel directions, this shows that the lines are parallel.

Non-Intersection of 2 Skew Lines

Example 34

Show that the lines $\frac{x - 12}{2} = \frac{y + 3}{-2} = \frac{z - 5}{4}$ and $\frac{x - 3}{1} = \frac{y + 7}{-3} = \frac{z - 1}{2}$ are skew.

For the lines to be skew, it must be shown that the direction vectors are not parallel and there is no common intersection point. Clearly, the direction vectors are not parallel. Equating the parametric equations gives,

$$2t - s = -9$$

$$-2t + 3s = -4$$

$$4t - 2s = -4$$

The first and third equations are clearly inconsistent. Hence, the lines are skew.

Intersections of Planes

Angle Between 2 Planes

The angle between 2 planes is taken to be the acute angle between the planes, and is equal to the acute angle made by their normals.

Example 35

Find the exact value (and the approximate value, to 1 d. p.) of the angle between the planes given by the equations $x + 2y + 2z = 5$ and $x - y + z = 5$.

The normals are $\begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$. The magnitudes of the normals are 3 and $\sqrt{3}$ respectively. Using the scalar product formula gives,

$$\begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} \bullet \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} = 3\sqrt{3}\cos\theta$$

$$\cos\theta = \frac{1}{3\sqrt{3}}$$

The acute angle thus has exact value $\cos^{-1}\left(\frac{1}{3\sqrt{3}}\right)$ and approximate value $78 \cdot 9\,°$.

Two planes are either coincident (the same), parallel or intersect in a line.

Intersection of 2 Planes in a Plane

If 2 planes are coincident, this is easily seen when the equations are written in Cartesian form. If they are written in vector form, more work is required.

<u>Example 36</u>

Show that the planes given by the vector equations
$\mathbf{r} = (1 + 2t)\,\mathbf{i} + (-2 + 3t + 2u)\,\mathbf{j} + (1 - t + u)\,\mathbf{k}$ and
$\mathbf{r} = (-1 + t)\,\mathbf{i} + (1 + t + u)\,\mathbf{j} + (5 - \tfrac{3}{4}t + \tfrac{1}{2}u)\,\mathbf{k}$ are the same.

If the planes are the same, there should be a common point and the normals must be parallel. For the common point, we need a unique solution for t and u to the equations,

$$1 + 2t = -1 + t$$

$$-2 + 3t + 2u = 1 + t + u$$

$$1 - t + u = 5 - \tfrac{3}{4}t + \tfrac{1}{2}u$$

The first equation gives $t = -2$. The second equation then gives $u = 7$. The third equation is satisfied (check !) by these values for t and u. Hence, the point $(-3, 6, 10)$ lies on both planes.

Rewriting the equation of the first plane as,

$$\mathbf{r} = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} + t\begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix} + u\begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}$$

shows that a normal to this plane is $\begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix} \times \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 5 \\ -2 \\ 4 \end{pmatrix}$. Similarly, a normal for the second plane,

$$\mathbf{r} = \begin{pmatrix} -1 \\ 1 \\ 5 \end{pmatrix} + t\begin{pmatrix} 1 \\ 1 \\ -\frac{3}{4} \end{pmatrix} + u\begin{pmatrix} 0 \\ 1 \\ \frac{1}{2} \end{pmatrix}$$

is $\begin{pmatrix} 1 \\ 1 \\ -\frac{3}{4} \end{pmatrix} \times \begin{pmatrix} 0 \\ 1 \\ \frac{1}{2} \end{pmatrix} = \begin{pmatrix} \frac{5}{4} \\ -\frac{1}{2} \\ 1 \end{pmatrix}$. The first normal is 4 times the second normal. Hence, as the normals are parallel vectors, and there exists a common point on both planes, the planes are identical.

Intersection of 2 Planes in a Line

<u>Example 37</u>

Find the line of intersection (in symmetric form) of the planes given by $x - 2y + 3z = 1$ and $2x + y + z = -3$.

The planes are not parallel to any of the x, y, or z axes so the planes will intersect at a point of the form $(x, y, 0)$ – alternative choices are to pick $(x, 0, z)$ or $(0, y, z)$ Hence, putting $z = 0$, we obtain the equations,

$$x - 2y = 1$$

$$2x + y = -3$$

which have solutions $x = -1$ and $y = -1$. Thus, $(-1, -1, 0)$ lies on the line. The vector product of the normals will give a vector in the same direction as the line (make a sketch !), and which can therefore be taken as the line direction. This normal direction is,

$$\begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix} \times \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -5 \\ 5 \\ 5 \end{pmatrix}$$

The line equation in symmetric form is thus,

$$\frac{x + 1}{-5} = \frac{y + 1}{5} = \frac{z}{5}$$

The other choices will obviously give different forms of the line equation.

Intersection of 3 Planes

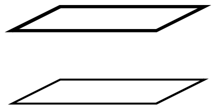
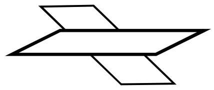
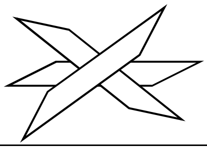
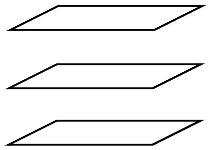
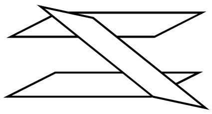
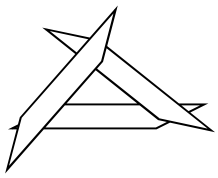
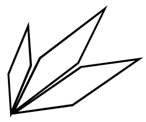
3 planes may intersect in 8 different ways. Only 4 of these give a common set of intersection points.

Before tabulating the possibilities, note that determining the intersection of 3 planes is a problem in solving a 3 x 3 system of equations. Also, if any 2 rows of the Augmented matrix are proportional (i.e., if one row is a multiple of the other), then the 2 planes represented by those equations are identical. If 2 rows of the Augmented matrix are not proportional, but the corresponding 2 rows of the Coefficient matrix are proportional, then the planes are parallel.

The possibilities will be listed by consideration of the proportionality of rows in the Augmented and Coefficient matrices.

The first column indicates initially how many rows of the Augmented matrix are proportional, the second how many rows of the Coefficient matrix are proportional, the third the row-reduced Augmented matrix, the fourth the number of solutions and the fifth the picture.

Note that number of solutions refers to a common set of intersection points *for all 3 planes*.

$\propto$ rows in AM	$\propto$ rows in CM	Row-reduced AM	Number of solutions	Picture
3	3	$\left(\begin{array}{ccc\|c} a & b & c & j \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$	Infinitely many points (in a plane)	
2	3	$\left(\begin{array}{ccc\|c} a & b & c & j \\ 0 & 0 & 0 & k \\ 0 & 0 & 0 & 0 \end{array}\right)$ $(k \neq 0)$	No solution (but 1 common plane of intersection)	
2	2	$\left(\begin{array}{ccc\|c} a & b & c & j \\ 0 & e & f & k \\ 0 & 0 & 0 & 0 \end{array}\right)$	Infinitely many points (in a line)	
2	2	$\left(\begin{array}{ccc\|c} a & b & c & j \\ 0 & e & f & k \\ 0 & 0 & 0 & 0 \end{array}\right)$	Infinitely many points (in a line)	
0	3	$\left(\begin{array}{ccc\|c} a & b & c & j \\ 0 & 0 & 0 & k \\ 0 & 0 & 0 & l \end{array}\right)$ $(k \neq 0,\ l \neq 0,\ k \neq l)$	No solution	
0	2	$\left(\begin{array}{ccc\|c} a & b & c & j \\ 0 & e & f & k \\ 0 & 0 & 0 & l \end{array}\right)$ $(l \neq 0)$	No solution (but 2 distinct intersection lines)	
0	2	$\left(\begin{array}{ccc\|c} a & b & c & j \\ 0 & e & f & k \\ 0 & 0 & 0 & l \end{array}\right)$ $(l \neq 0)$	No solution (but 3 distinct intersection lines)	
0	0	$\left(\begin{array}{ccc\|c} a & b & c & j \\ 0 & e & f & k \\ 0 & 0 & i & l \end{array}\right)$	1 point	

Intersections of Lines and Planes

A line and a plane can intersect in either a single point or in a whole line.

Intersection of a Line and a Plane in a Point

Example 38

Find the point of intersection of the line $\frac{x - 7}{3} = \frac{y - 11}{4} = \frac{z - 24}{13}$ and the plane $6x + 4y - 5z = 28$.

The parametric line equations are,

$$x = 7 + 3t, \quad y = 11 + 4t, \quad z = 24 + 13t$$

Substituting these into the plane equation gives (check !) $t = -2$. Substituting this value of t back into the parametric equations gives the intersection point as $(1, 3, -2)$.

Intersection of a Line and a Plane in a Line

Example 39

Show that the line $\frac{x - 3}{12} = \frac{y + 4}{-4} = \frac{z - 1}{3}$ lies in the plane $2x + 3y - 4z = -10$.

If the line lies in the plane, then the line direction and plane normal must be perpendicular.

$$\begin{pmatrix} 12 \\ -4 \\ 3 \end{pmatrix} \bullet \begin{pmatrix} 2 \\ 3 \\ -4 \end{pmatrix} = 24 - 12 - 12 = 0$$

Hence, the line lies in the plane.

Example 40

Find the angle made by the line $\frac{x - 7}{3} = \frac{y - 11}{4} = \frac{z - 24}{13}$ and the plane $6x + 4y - 5z = 28$.

The required angle is just the complement of the angle between the line direction and the plane normal (make a sketch). The angle between the normal and direction is found, as usual, by the scalar product formula and turns out to be (check !) $104 \cdot 7°$, or as we take the acute angle, $75 \cdot 3°$. Thus, the required angle is $14 \cdot 7°$. Make a sketch for clarification !

Distances of Points from Lines and Planes

Distance From a Point to a Plane

The distance from a point to a plane is the shortest (i.e. perpendicular) distance from the point to the plane.

Example 41

Find the distance of P (1, 2, 1) from the plane $x - y + 2z = 5$.

The line joining P to the point Q which is at the shortest distance from P has direction vector parallel to the plane normal. Hence, the equation of this line is,

$$\frac{x - 1}{1} = \frac{y - 2}{-1} = \frac{z - 1}{2} = t$$

Writing these equations parametrically and substituting into the plane equation gives $t = 2/3$ and thus the coordinates of Q as (5/3, 4/3, 7/3). The distance formula (in 3D) then gives the distance PQ as $\frac{2\sqrt{6}}{3}$.

Distance From a Point to a Line

The distance from a point to a line is the shortest (i.e. perpendicular) distance from the point to the plane.

Example 42

Find the distance of P (15, −9, −2) from the line

$$\frac{x - 35}{8} = \frac{y + 43}{-10} = \frac{z - 62}{13} = t\,.$$

The line joining P to the point Q which is at the shortest distance from P has direction vector perpendicular to the given line direction. So, Q has coordinates $(35 + 8t, -43 - 10t, 62 + 13t)$. Thus, the vector joining P to Q has coordinates $\begin{pmatrix} 20 + 8t \\ -34 - 10t \\ 64 + 13t \end{pmatrix}$. As this vector is perpendicular to the line direction,

$$\begin{pmatrix} 20 + 8t \\ -34 - 10t \\ 64 + 13t \end{pmatrix} \bullet \begin{pmatrix} 8 \\ -10 \\ 13 \end{pmatrix} = 0$$

implies that $t = -4$. Hence, Q has coordinates (3, 3, 10) and the required distance works out to be 18 units.