Maclaurin Series Questions

1. (a) Given that $f(x) = \sin(3x + \frac{\pi}{4})$ use Maclaurin's series up to the term in x^2 to show that

$$f(x) \approx \frac{\sqrt{2}}{2}(1 + 3x - \frac{9}{2}x^2) \qquad \text{for small values of } x.$$

(b) Hence, or otherwise, obtain a series approximation up to the term in x^2 for

$$e^x \sin(3x + \frac{\pi}{4})$$

2. The function f is given by $f(x) = \ln(1 - 2x)$.

(a) (i) Find $f'(x)$, $f''(x)$ and $f'''(x)$.

(ii) Hence show that the first three terms of the Maclaurin series for $\ln(1 - 2x)$ are

$$-2x - 2x^2 - \frac{8x^3}{3}.$$

(b) Show that for small values of x

$$2xe^x + \ln(1 - 2x) \approx kx^3,$$

where the value of the constant k is to be stated.

3. (a) (i) Given that

$$f(x) = \cos 2x,$$

find $f'(x)$ and $f''(x)$.

(ii) Hence, using the Maclaurin series, show that

$$1 - 2x^2$$

is a quadratic approximation for $\cos 2x$ for small values of x.

(b) Given that, for small values of x,

$$\sin x \approx x \qquad \text{and} \qquad e^x \approx 1 + x + \frac{1}{2}x^2$$

obtain a quadratic approximation for

$$e^{-2x} + \sin x.$$

(c) Hence obtain an approximate solution to the equation

$$e^{-2x} + \sin x = \cos 2x, \quad x > 0.$$

4. The function f is given by $f(x) = \dfrac{1}{3x - 2}$.

(a) (i) Differentiate the function f twice to find $f'(x)$ and $f''(x)$.

(ii) Hence find the Maclaurin series for $f(x)$ up to the term in x^2.

(b) Given that

$$\frac{(x-1)(x-2)}{x^2(3x-2)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{(3x-2)},$$

find the values of B and C and show that $A = 0$.

(c) Use your results from parts (a) and (b) to show that, for small values of x,

$$\frac{(x-1)(x-2)}{(3x-2)} \approx -1 - \frac{1}{2}x^2 - \frac{3}{4}x^3 - \frac{9}{8}x^4.$$

5. (a) (i) The function f is given by

$$f(x) = e^{-2x}.$$

By differentiation, find $f'(x)$ and $f''(x)$

(ii) Hence show that the first three terms in the Maclaurin series of $f(x)$ are

$$1 - 2x + 2x^2$$

(b) (i) Using the approximation $\cos x \approx 1 - \frac{x^2}{2}$, write down a similar approximation for $\cos 3x$.

(ii) Use your results from parts (a)(ii) and (b)(i) to find an approximate solution of the equation

$$e^{-2x} = \cos 3x \qquad \text{for } 0 < x < 1.$$

6. The function f is given by $f(x) = \sin\left(2x + \frac{\pi}{6}\right)$.

(a) (i) Find $f'(x)$ and $f''(x)$

(ii) Hence, using the Maclaurin series, show that, for small values of x,

$$f(x) \approx \frac{1}{2} + \sqrt{3}x - x^2$$

(b) Show that, for small values of x,

$$(1 - \cos x)\sin\left(2x + \frac{\pi}{6}\right) \approx kx^2$$

where the value of the constant k is to be found.

7. (a) The function f is given by $f(x) = e^{-3x}$

(i) Find $f'(x)$ and $f''(x)$.

(ii) Hence show that the first three terms of the Maclaurin series of $f(x)$ are

$$1 - 3x + \frac{9}{2}x^2.$$

(b) Use the approximation $\ln(1 + x) \approx x - \frac{x^2}{2} + \frac{x^3}{3}$ to obtain a similar approximation for $\ln(1 + 3x)$ for small values of x.

(c) Use the results for parts (a) and (b) to estimate the positive value of x for which

$$\ln(1 + 3x) - 2xe^{-3x} = 0.1.$$

Give your value of x to three decimal places.

Solutions

1. (b) $g(x) = \frac{\sqrt{2}}{2}(1 + 4x - x^2)$

2. (a) (i) $f(x) = \ln(1 - 2x)$

$f'(x) = -2(1 - 2x)^{-1}$

$f''(x) = -4(1 - 2x)^{-2}$

$f'''(x) = -16(1 - 2x)^{-3}$

(b) $k = \frac{-5}{3}$

3. (a) (i) $f(x) = \cos 2x$

$f'(x) = -2 \sin 2x$

$f''(x) = -4 \cos 2x$

(b) $1 - x + 2x^2$

(c) $x = \frac{1}{4}$

4. (a) (i) $f(x) = (3x - 2)^{-1}$

$f'(x) = -1 \times 3(3x - 2)^{-2}$

$f''(x) = 18\,(3x - 2)^{-3}$

(ii) $-\frac{1}{2} - \frac{3}{4}x - \frac{9}{8}x^2$

(b) $B = -1$, $C = 1$

5. (a) (i) $f(x) = e^{-2x}$ $f'(x) = -2e^{-2x}$

$f''(x) = 4e^{-2x}$

(b) (i) $\cos 3x \approx 1 - \frac{(3x)^2}{2}$

(ii) $x = \frac{4}{13}$

6. (a) (i) $f(x) = \sin\left(2x + \frac{\pi}{6}\right)$, $f'(x) = 2\cos\left(2x + \frac{\pi}{6}\right)$

$f''(x) = -4\sin\left(2x + \frac{\pi}{6}\right)$

(b) $k = \frac{1}{4}$

7. (a) (i) $f(x) = e^{-3x}$

$f'(x) = -3e^{-3x}$

$f''(x) = 9e^{-3x}$

(b) $3x - \frac{9}{2}x^2 + 9x^3$

(c) 0.088