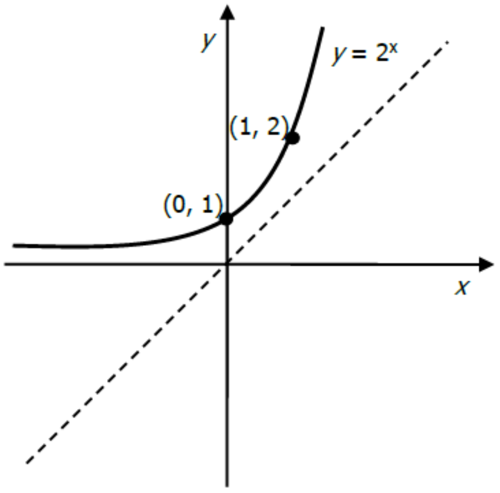
Exponential functions have the formula $f(x) = a^x$, $x \in \mathbf{R}$, where a is called the **base**.
The graph of $y = 2^x$ is shown below.



The graph of $y = 2^x$ passes through the points (0, 1) and (1, 2). As reflection in the line y = x will produce the inverse of $y = 2^x$, then the inverse of $f(x) = 2^x$ must pass through the points (1, 0) and (2, 1)

The inverse of an exponential function is known as a **logarithmic** function.

If $f(x) = a^x$, then $f^{-1}(x) = \log_a x$
("log to the base a of x")

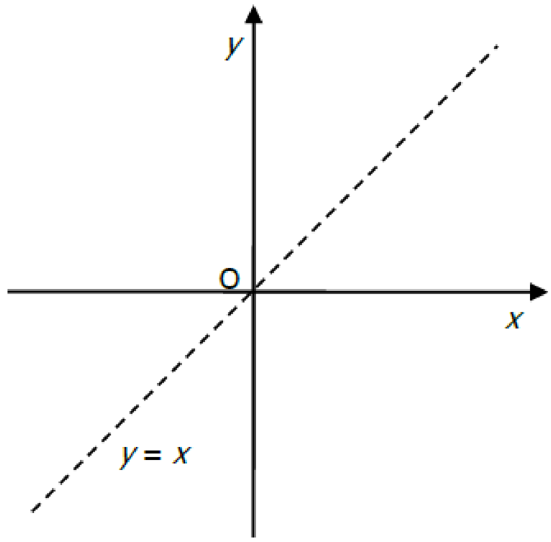
Example 5: Add the graph of $y = \log_2 x$ to the graph opposite.

Note that:

$y = a^x$ passes through (0, 1) and (1, a)
$y = \log_a x$ passes through (1, 0) and (a, 1)

For logarithms:

If $y = a^x$
then
$\log_a y = x$



Example 6: On the graph above, sketch and annotate the graphs of:

a) $y = 5^x$

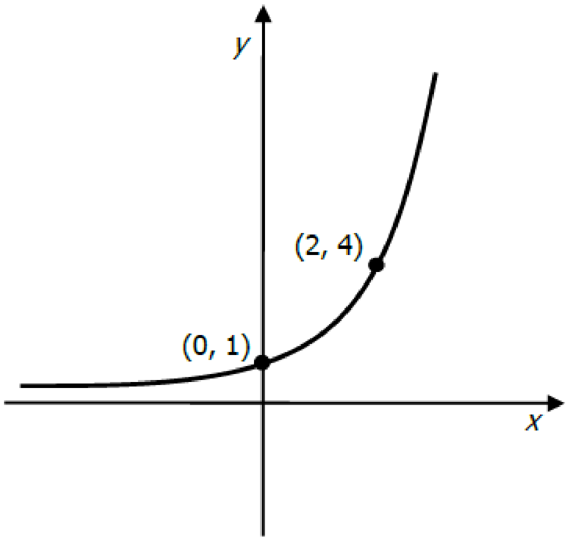
b) $y = \log_5 x$

Example 7: Write as logarithms:

a) $y = 3^x$

b) $q = 13^r$

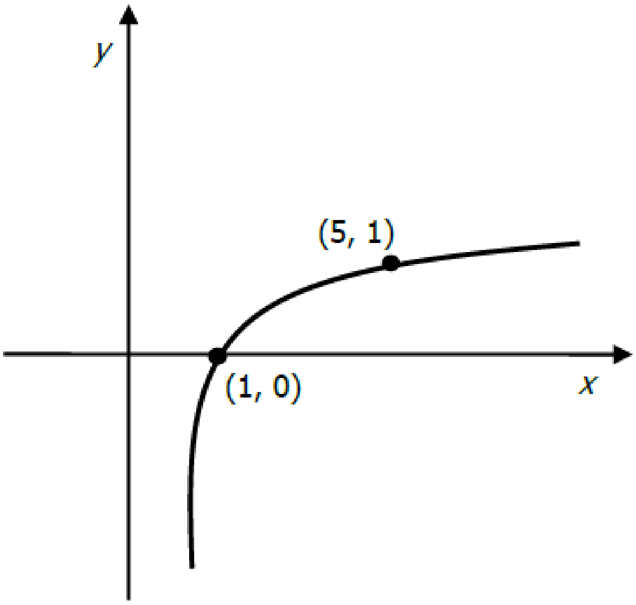
$y = a^x$ means "a multiplied by itself x times gives y"
$y = \log_a x$ means "y is the number of times I multiply a by itself to get x"



Example 8: Shown is the graph of the function $f(x) = 2^x$. To the diagram opposite, add the annotated graphs of the functions:

a) $y = 2^x - 3$

b) $y = 2^{(x - 2)}$



Example 9: Shown is the graph of the function $y = \log_5 x$. To the diagram opposite, add the annotated graphs of the functions:

a) $y = 2\log_5 x$

b) $y = \log_5(x + 1)$

Past Paper Example:

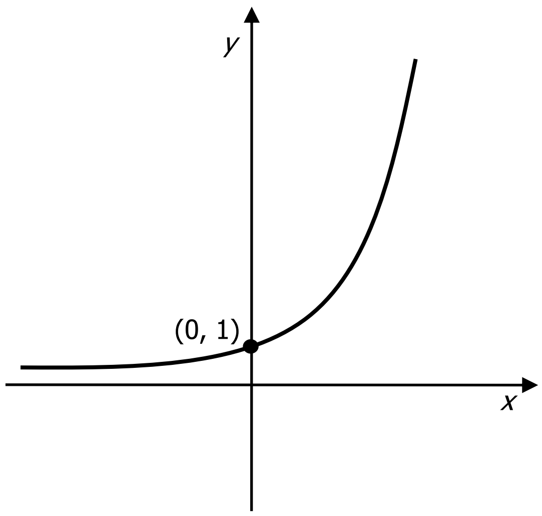
Functions f and g are defined on the set of real numbers. The inverse functions f^{-1} and g^{-1} both exist.

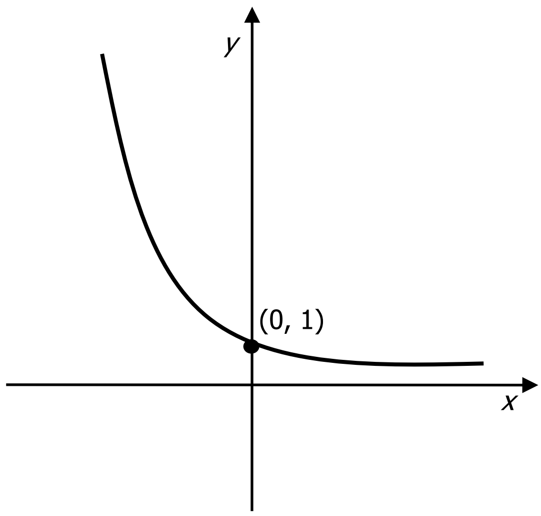
a) Given $f(x) = 3x + 5$, find $f^{-1}(x)$

b) If $g(2) = 7$, write down the value of $g^{-1}(7)$.

Exponential and Logarithmic Functions

Exponential functions are those with variable powers, e.g. $y = a^x$. Their graphs take two forms:





When a > 1, the graph:

- is always increasing
- is always positive
- never cuts the *x* – axis
- passes through (0, 1)
- shows **exponential growth**

When 0 < a < 1, the graph

- is always decreasing
- is always positive
- never cuts the *x* – axis
- passes through (0, 1)
- shows **exponential decay**

Exponential Functions as Models

Example 11: Ulanda's population in 2016 was 100 million and it was growing at 6% per annum.

a) Find a formula P_n for the population in millions, n years later.

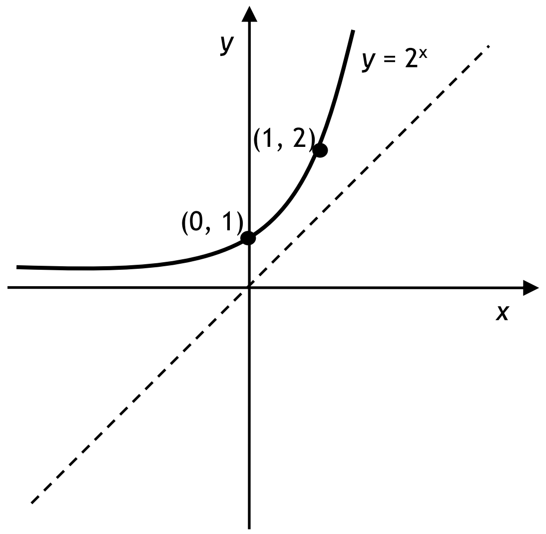
b) Estimate the population in the year 2026

Example 12: 8000 gallons of oil are lost in an oil spill in Blue Sky Bay. At the beginning of each week a filter plant removes 67% of the oil present.

a) Find a formula G_n for the amount of oil left in the bay after n weeks.

b) After how many **complete** weeks will there be less than 10 gallons left?

Logarithmic Functions



The inverse of an exponential function is known as a **logarithmic** function.

If $f(x) = a^x$, then $f^{-1}(x) = \log_a x$

("log to the base a of x")

We have seen that the graph of the inverse of a function can be obtained by reflection in the line $y = x$.

Since the graph of $y = 2^x$ passes through the points (0, 1) and (1, 2), then the inverse of $f(x) = 2^x$ must pass through the points (1, 0) and (2, 1).

Example 13: Add the graph of $y = \log_2 x$ to the graph opposite.

Note that:

$y = a^x$ passes through (0, 1) and (1, a)

$y = \log_a x$ passes through (1, 0) and (a, 1)

$y = a^x$ means "a multiplied by itself x times gives y"

$y = \log_a x$ means "y is the number of times I multiply a by itself to get x"

Since the graph does not cross the y-axis, we can only take the logarithm of a positive number

The expression "$\log_a x$ "can be read as "a to the power of what is equal to x ?", e.g. $\log_2 8$ means "2 to the power of what equals 8?", so $\log_2 8 = 3$.

Example 14: Write in logarithmic form:

a) $5^2 = 25$

b) $12^1 = 12$

c) $8^{1/3} = 2$

d) $8^x = y$

e) $1 = q^0$

f) $(x - 3)^4 = k$

Example 15: Write in exponential form:

a) $3 = \log_5 125$

b) $\log_7 49 = 2$

c) $\log_4 4096 = 6$

d) $\log_2\left(\frac{1}{4}\right) = -2$

e) $\log_b g = 5h$

f) $1 = \log_7 7$

Example 16: Evaluate:

a) $\log_8 64$

b) $\log_2 32$

c) $\log_{3.5} 3.5$

d) $\log_{25} 5$

e) $\log_4\left(\frac{1}{2}\right)$

Since $a^1 = a$, then **$\log_a a = 1$**

Since $a^0 = 1$, then **$\log_a 1 = 0$.**

Laws of Logarithms

$\log_a xy = \log_a x + \log_a y$	$\log_a\left(\frac{x}{y}\right) = \log_a x - \log_a y$	$\log_a x^n = n\log_a x$

Example 17:

a) $\log_2 4 + \log_2 8 - \log_2\frac{1}{2}$

b) $2\log_5 10 - \log_5 4$

c) Simplify $\frac{1}{4}(\log_3 810 - \log_3 10)$

Solving Logarithmic Equations

You **MUST** memorise the laws of logarithms to solve log equations! As we can only take logs of **positive** numbers, we must remember to discard any answers which violate this rule!

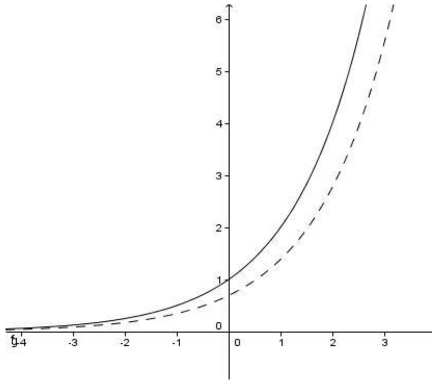
Example 18: Solve:

a) $\log_4(3x - 2) - \log_4(x + 1) = \frac{1}{2}$ $\left(x > \frac{2}{3}\right)$

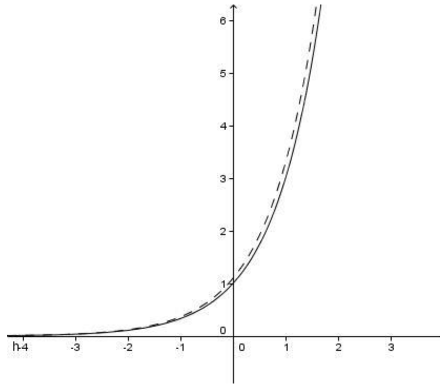
b) $\log_6 x + \log_6(2x - 1) = 2$ $\left(x > \frac{1}{2}\right)$

The Exponential Function and Natural Logarithms

The graph of the derived function of $y = a^x$ can be plotted and compared with the original function. The new graphs are also exponential functions. Below are the graphs of $y = 2^x$ and $y = 3^x$ (solid lines) and their derived functions (dotted).



$f(x) = 2^x$



$f(x) = 3^x$

The derived function of $y = 2^x$ lies **under** the original graph, but the derived function of $y = 3^x$ lies **above** it.

This means that there must be a value of a between 2 and 3 where the derived function lies **on** the original.

i.e. where $f(x) = f'(x)$

The value of the base of this function is known as e, and is approximately 2.71828.

The function $y = e^{x}$ is known as The Exponential Function.

The function $y = \log_e x$ is known as the Natural Logarithm of x, and is also written as $\ln x$.

Example 19: Evaluate:

a) e^3

b) $\log_e 120$

Example 20: Solve:

a) $\ln x = 5$

b) $5^{x-1} = 16$

Example 21: Atmospheric pressure P_t at various heights above sea level can be determined by using the formula $P_t = P_0 e^{rt}$, where P_0 is the pressure at sea level, t is the height above sea level in thousands of feet, and r is a constant.

a) At 20 000 feet, the air pressure is half that at sea level. Find r accurate to 3 significant figures.

b) Find the height at which P is 10% of that at sea level.

Example 22: A radioactive element decays according to the law $A_t = A_0 e^{kt}$, where A_t is the number of radioactive nuclei present at time t years and A_0 is the initial amount of radioactive nuclei.

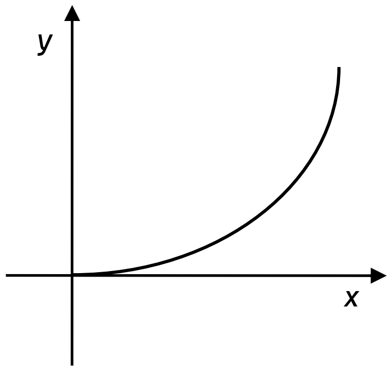
a) After 150 years, 240g of this material had decayed to 200g.
Find the value of k accurate to 3 s.f.

b) The half-life of the element is the time taken half the mass to decay. Find the half-life of the material.

Using Logs to Analyse Data, Type 1: $y = kx^n \Leftrightarrow \log y = n \log x + \log k$

When the data obtained from an experiment results in an exponential graph of the form $y = kx^n$ as shown below, we can use the laws of logarithms to find the values of k and n.

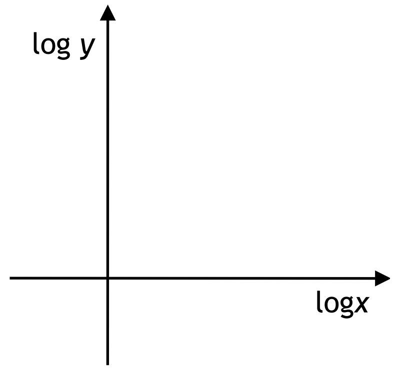
To begin, take logs of both sides of the exponential equation.



$y = kx^n$

$y = kx^n$

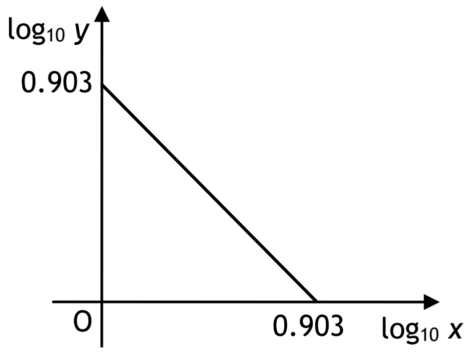
This gives a straight line graph!



$\log y = n \log x + \log k$

Note: the base is not important, as long as the **same** base is used on both sides.

Example 23: Data are recorded from an experiment and the graph opposite is produced.



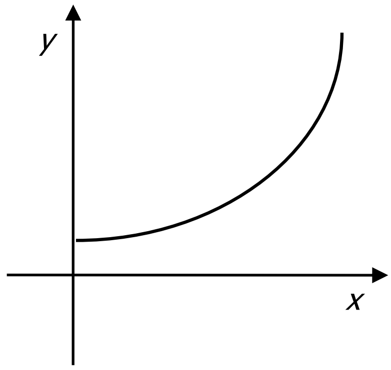
a) Find the equation of the line in terms of $\log_{10} x$ and $\log_{10} y$.

b) Hence express y in terms of x.

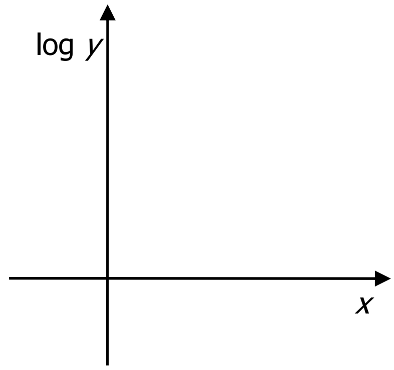
Using Logs to Analyse Data, Type 2: $y = kn^x \Leftrightarrow \log y = \log n\ (x) + \log k$

A similar technique can be used when the graph is of the form $y = kn^x$ (i.e. x is the index, not the base as before).

$y = kn^x$

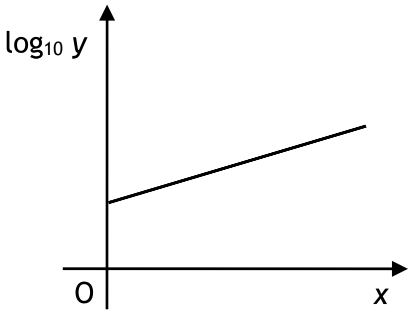


$y = kn^x$



$\log y = (\log n)\, x + \log k$

Example 24: The data below are plotted and the graph shown is obtained.

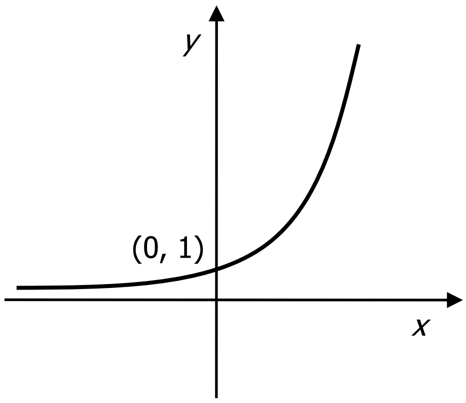
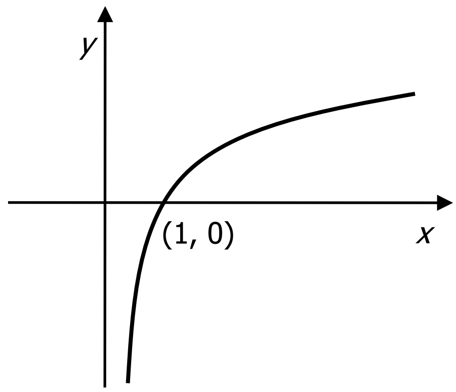


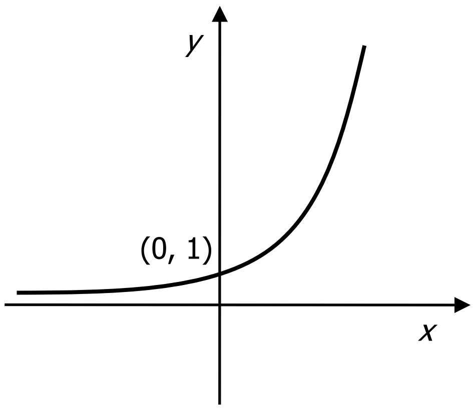
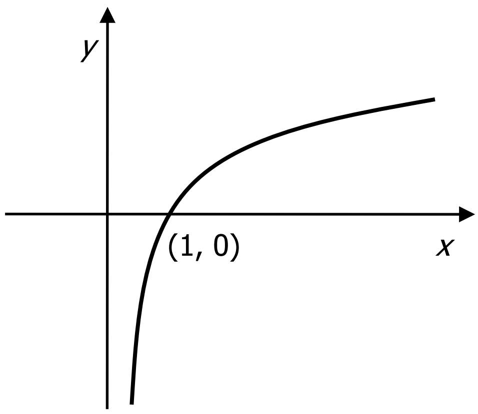
x	0	1	2	3	4
$\log_{10} y$	0.602	1.079	1.556	2.033	2.510

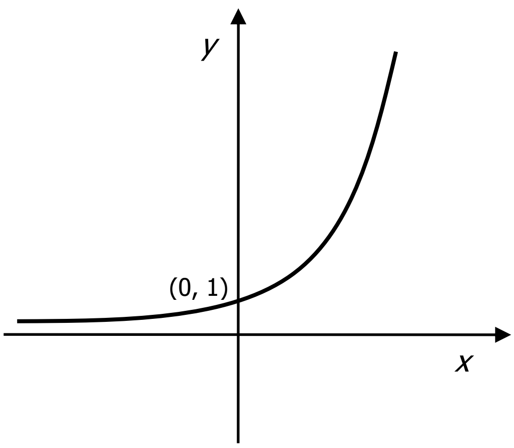
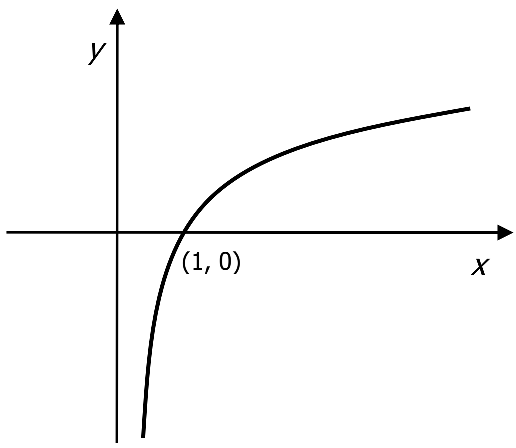
a) Express $\log_{10} y$ in terms of x.

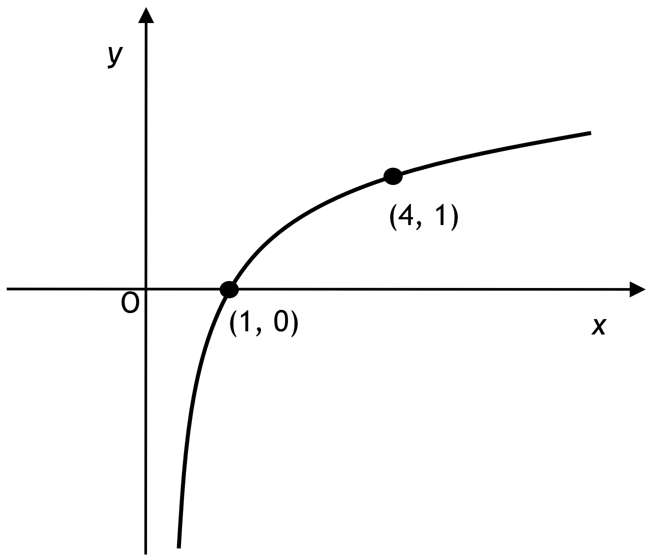
b) Hence express y in terms of x.

Related Graphs of Exponentials and Logs

$y = e^x + a$	$y = \ln(x + a)$
	
$y = e^x + a$ is obtained by **sliding** $y = e^x$: **Vertically upwards if a>0** **Vertically downwards if a<0**	$y = \ln(x + a)$ is obtained by **sliding** $y = \ln x$ **Horizontally left if a>0** **Horizontally right if a<0**

$y = e^{(x + a)}$	$y = k \ln x$
	
$y = e^{(x + a)}$ is obtained by **sliding** $y = e^x$ **Horizontally left if a>0** **Horizontally right if a<0**	$y = k \ln x$ is obtained by **vertically:** **stretching** $y = \ln x$ **if** $k > 1$ **compressing** $y = \ln x$ **if** $0 < k < 1$

$y = e^{-x}$	$y = - \ln x$
	
$y = e^{-x}$ is obtained by **reflecting** $y = e^x$**:** **in the** y **-axis**	$y = - \ln x$ is obtained by **reflecting** $y = \ln x$: **in the** x **- axis**



Example 25: The graph of $y = \log_4 x$ is shown. On the same diagram, sketch:

a) $y = \log_4 4x$

b) $y = \log_4 \left(\frac{1}{4x}\right)$

Past Paper Example 1:

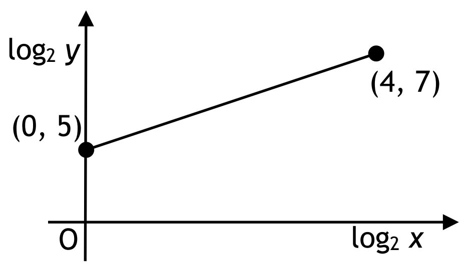
a) Show that $x = 1$ is a root of $x^3 + 8x^2 + 11x - 20 = 0$, and hence factorise $x^3 + 8x^2 + 11x - 20$ fully

b) Solve $\log_2(x + 3) + \log_2(x^2 + 5x - 4) = 3$

Past Paper Example 2: Variables x and y are related by the equation $y = kx^n$.

The graph of $\log_2 y$ against $\log_2 x$ is a straight line through the points (0, 5) and (4, 7), as shown in the diagram.

Find the values of k and n.



Past Paper Example 3: The concentration of the pesticide *Xpesto* in soil is modelled by the equation:

$$P_t = P_0 e^{-kt}$$

where: P_0 is the initial concentration
P_t is the concentration at time t
t is the time, in days, after the application of the pesticide.

a) Once in the soil, the half-life of a pesticide is the time taken for its concentration to be reduced to one half of its initial value.

If the half-life of *Xpesto* is 25 days, find the value of k to 2 significant figures.

b) Eighty days after the initial application, what is the percentage decrease in *Xpesto*?

Past Paper Example 4: Simplify the expression $3\log_e 2e - 2\log_e 3e$ giving your answer in the form $A + \log_e B - \log_e C$, where A, B and C are whole numbers.

Past Paper Example 5: Two variables x and y satisfy the equation $y = 3(4^x)$.

A graph is drawn of $\log_{10} y$ against x. Show that its equation will be of the form $\log_{10} y = Px + Q$, and state the gradient and y-intercept of this line.