

# Logs and Exponentials

Higher Mathematics Supplementary Resources

## Section A

This section is designed to provide examples which develop routine skills necessary for completion of this section.

### R1 I have had experience of simplifying expressions with surds and indices.

1. Simplify the following writing the answers with positive indices only.

(a)  $x^2 \times x^7$                       (b)  $y^{-3} \times y^8$                       (c)  $x^6 \div x^4$

(d)  $y^{-3} \div y^{-1}$                       (e)  $(a^4)^3$                       (f)  $(p^{-4})^{-2}$

2. (a)  $4x^3 \times 2x^2$                       (b)  $5x^3 \times 4x^{-3}$                       (c)  $\frac{3x^5y^3}{6x^2y^5}$

(d)  $\frac{4r^8}{2r^{-5}}$                       (e)  $\sqrt{x} \times x^2$                       (f)  $\sqrt{x} \times \sqrt[3]{x^2}$

(g)  $\frac{1}{\sqrt{a}} \times \sqrt{a^3}$                       (h)  $\sqrt[3]{x} \times x^{-\frac{1}{3}}$                       (i)  $x^{-2}(x^2 + 1)$

### R2 I can write an exponential in logarithmic form and vice versa.

1. For each exponential relationship, write a related logarithmic relationship.

(a)  $3^x = 5$                       (b)  $8^t = 100$                       (c)  $5^r = 13$

(d)  $10^r = 5$                       (e)  $6^s = 15$                       (f)  $2^p = 32$

(g)  $a^x = 5$                       (h)  $b^y = 5$                       (i)  $c^z = 5$

(j)  $2^3 = 8$                       (k)  $3^2 = 9$                       (l)  $10^4 = 10000$

(m)  $a^b = c$                       (n)  $x^y = z$                       (o)  $p^q = r$

2. For each logarithmic relationship, write a related exponential relationship.

(a)  $\log_x 20 = 3$                       (b)  $\log_e x = 2$                       (c)  $\log_3 7 = x$

(d)  $\log_y 16 = 4$                       (e)  $\log_e r = 1 \cdot 2$                       (f)  $\log_6 9 = t$

<b>(g)</b> $\log_3 x = 2$	<b>(h)</b> $\log_{10} y = 2 \cdot 4$	<b>(i)</b> $\log_{(e+1)} R = v$
<b>(j)</b> $\log_{(x+2)} T = r$	<b>(k)</b> $\log_5 r = x + 1$	<b>(l)</b> $\log_e x = r - 2$

**R3 I can solve exponential equations using logarithms.**

1. Solve each of the following exponential equations

<b>(a)</b> $10^x = 1000$	<b>(b)</b> $10^x = 0 \cdot 01$	<b>(c)</b> $10^x = 100000$
<b>(d)</b> $10^x = 20$	<b>(e)</b> $10^x = 3000$	<b>(f)</b> $10^x = 0 \cdot 05$
<b>(g)</b> $10^{2p} = 5$	<b>(h)</b> $10^{9t} = 500$	<b>(i)</b> $10^{-4r} = 0 \cdot 6$
<b>(j)</b> $10^{-3p} = 20$	<b>(k)</b> $10^{0 \cdot 1y} = 3000$	<b>(l)</b> $10^{0 \cdot 5q} = 0 \cdot 05$

2. Solve each of the following exponential equations

<b>(a)</b> $e^x = 7$	<b>(b)</b> $e^x = 23$	<b>(c)</b> $e^{2t} = 9$
<b>(d)</b> $e^{3p} = 16$	<b>(e)</b> $e^{-3x} = 0 \cdot 4$	<b>(f)</b> $e^{-9r} = 1 \cdot 3$
<b>(g)</b> $6e^x = 12$	<b>(h)</b> $4e^{0 \cdot 02x} = 12$	<b>(i)</b> $12e^{3t} = 6$
<b>(j)</b> $3e^{-4p} = 21$	<b>(k)</b> $2e^{-2x} = 0 \cdot 4$	<b>(l)</b> $6e^{-0 \cdot 5r} = 1 \cdot 3$

3. Solve each of the following exponential equations

<b>(a)</b> $4 \times 10^x = 400$	<b>(b)</b> $7 \times 10^p = 0 \cdot 67$	<b>(c)</b> $3 \times 10^t = 12$
<b>(d)</b> $0 \cdot 5 \times 10^a = 20$	<b>(e)</b> $\frac{10^b}{8} = 12 \cdot 5$	<b>(f)</b> $\frac{10^c}{100} = 0 \cdot 05$

## Section B

This section is designed to provide examples which develop Course Assessment level skills

**NR1 I can manipulate logarithms and exponentials and apply the three main laws of logarithms.**

- Given  $b = e^t$  which of the following is true:
  - $\log_t b = e$
  - $\log_e b = t$
- Given  $\log_n x = y$  which of the following is true:
  - $n^y = x$
  - $x^y = n$
- Simplify
  - $\log_x 3 + \log_x 5 - \log_x 7$
  - $\log_a 32 - 2\log_a 4$
- Show that
  - $\frac{\log_3 8}{\log_3 2} = 3$  .
  - $\frac{\log_b 9a^2}{\log_b 3a} = 2$  .
- If  $\log_3 x = 2\log_3 y - 3\log_3 z$  find an expression for  $x$  in terms of  $y$  and  $z$ .
- Find  $a$  if  $\log_a 64 = \frac{3}{2}$  .
- Simplify  $2\log_e(3e) - 4\log_e(2e)$  expressing your answer in the form  $\log_e B - \log_e C - A$  where  $A, B$  and  $C$  are whole numbers.

**NR2 I can solve exponentials and logarithmic equations using the laws of logarithms.**

1. Given the equation  $y = 3 \times 4^x$  find the value of  $x$  when  $y = 10$  giving your answer to 3 significant figures.
  
2. Given the equation  $A = A_0 e^{-kt}$ , find, to 3 significant figures:
  - (a)  $A$  when  $A_0 = 5$ ,  $k = 0.23$  and  $t = 20$ .
  - (b)  $k$  when  $A = 70$ ,  $A_0 = 35$  and  $t = 20$ .
  - (c)  $t$  when  $A = 1000$ ,  $A_0 = 10$  and  $k = -0.01$ .
  
3.
  - (a) Solve the equation  $\log_3(3 - 2x) + \log_3(2 + x) = 1$ .
  - (b) Solve  $\log_3(5 - x) - \log_3(3 - x) = 2$ ,  $x < 3$ .
  - (c) Solve  $\log_4 x + \log_4(x + 6) = 2$ .
  
4.
  - (a) Given that  $\log_5 x = A$ , show that  $\log_{25} x = \frac{1}{2}A$ .
  - (b) Solve  $\log_4 x + \log_{16} x = 3$ .
  
5. The curve with equation  $y = \log_4(x - 1) - 2$ , where  $x > 1$ , cuts the x-axis at the point  $(p, 0)$ . Find the value of  $p$ .
  
6. If  $\log_4 8 + \log_4 q = 1$ , find the value of  $q$ .
  
7. Solve the equation  $\log_3(x + 2) - 3\log_3 2 = 2$ .
  
8. Find  $x$  if  $4\log_x 6 - 2\log_x 4 = 1$ .
  
9. Find the x-coordinate of the point where the graph of the curve with equation  $y = \log_3(x - 2) + 1$  intersects the x-axis.

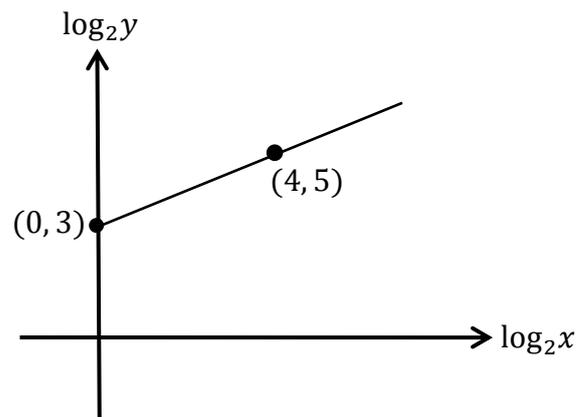
**NR3 I have experience of plotting and extracting information from straight line graphs with logarithmic axes (axis).**

1. Show that  $y = kx^n$ , where  $k$  and  $n$  are constants, can be expressed as a straight line in terms of  $\log y$  and  $\log x$ .
2. Show that  $y = Ae^{kx}$ , where  $k$  and  $n$  are constants, can be expressed as a straight line in terms of  $\log y$  and  $x$ .

3. Variables  $x$  and  $y$  are related by the equation  $y = kx^n$

The graph of  $\log_2 y$  against  $\log_2 x$  is a straight line through the points  $(0, 3)$  and  $(4, 5)$ , as shown in the diagram.

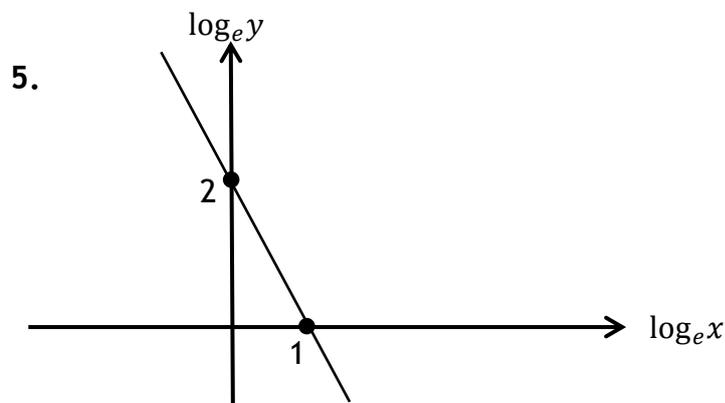
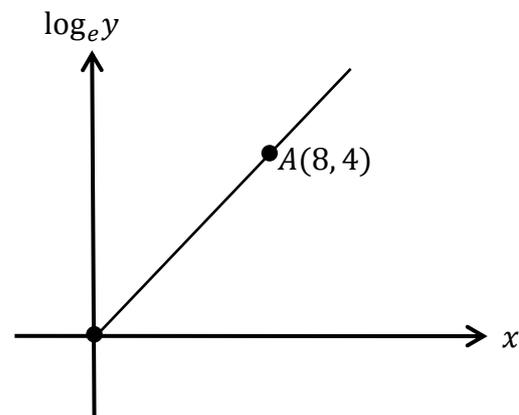
Find the values of  $k$  and  $n$ .



4. Two variables,  $x$  and  $y$ , are connected by the law  $y = a^x$ .

The graph of  $\log_e y$  against  $x$  is a straight line passing through the origin and the point  $A(8, 4)$ .

Find the value of  $a$ .

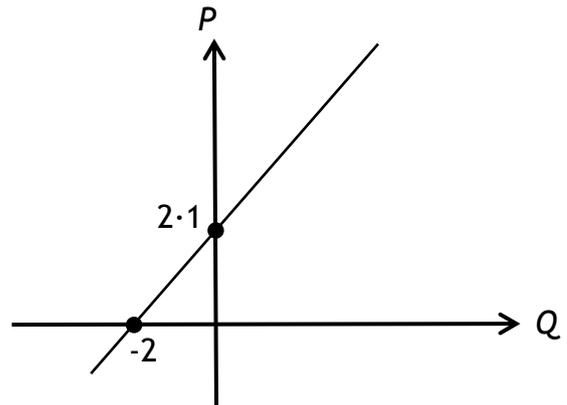


The graph illustrates the law  $y = kx^n$ .

If the straight line passes through  $A(1, 0)$  and  $B(0, 2)$ , find the values of  $k$  and  $n$ .

6. The result of an experiment gives rise to the graph shown.

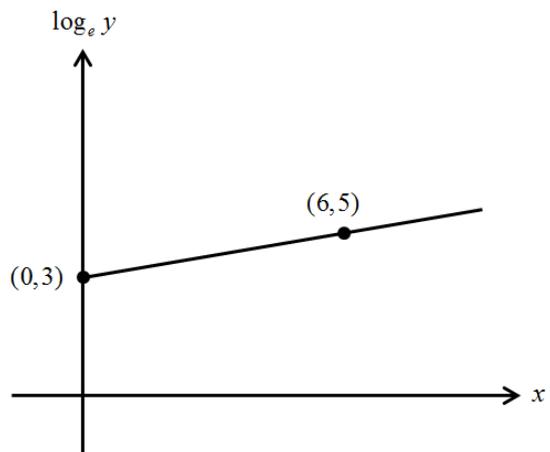
- (a) Write down the equation of the line in terms of  $P$  and  $Q$ .
- (b) If  $P = \log_{10} p$  and  $Q = \log_{10} q$ , show that  $p$  and  $q$  satisfy  $p = aq^b$  stating the values of  $a$  and  $b$ .



7. Variables  $x$  and  $y$  are related by the equation  $y = Ae^{kx}$

The graph of  $\log_e y$  against  $x$  is a straight line through the points  $(0, 3)$  and  $(6, 5)$ , as shown in the diagram.

Find the values of  $k$  and  $A$ .



8. Two variables  $x$  and  $y$  satisfy the equation  $y = 5 \times 4^x$ .

- (a) Find the value of  $a$  if  $(a, 10)$  lies on the graph with equation  $y = 5 \times 4^x$ .
- (b) If  $(-\frac{1}{2}, b)$  also lies on the graph, find  $b$ .
- (c) A graph is drawn of  $\log_{10} y$  against  $x$ . Show that its equations will be of the form  $\log_{10} y = Px + Q$  and state the gradient of this line.

**NR4 I can solve logarithmic and exponential equations in real life contexts.**

1. The amount of a certain radioactive element,  $A_t$ , remaining after  $t$  years can be found using the formula  $A_t = A_0 e^{-0.002t}$ , where  $A_0$  is the amount present initially.
  - (a) If 300 grams are left after 500 years, how many grams were present initially.
  - (b) The half-life of a substance is the time taken for the amount to decrease to half its initial amount. What is the half life of this substance?
  
2. The value  $V$  (in £ thousand) of a car is shown to depreciate after  $t$  years from first purchase according to the formula  $V = 18e^{-0.15t}$ .
  - (a) What was the value of the car when first purchased?
  - (b) The car was sold when its value had dropped to 10% of the value when first purchased.

After how many years was the car sold?
  
3. The formula  $A_t = A_0 e^{-0.000124t}$  is used to determine the age of wood, where  $A_0$  is the amount of carbon-14 in any living tree,  $A_t$  is the amount of carbon-14 in the wood being dated and  $t$  is the age of the wood in years.

A wooden artefact was found to contain 90% of the carbon-14 of a living tree.

Is the artefact over 500 years old?

4. The size of a rabbit population,  $N$ , can be modelled using the equation  $N = N_0 e^{kt}$  where  $N_0$  is the population at the beginning of a study and  $t$  is the time in years since the study began and  $k$  is a constant.

- (a) The rabbit population comprised of 70 individuals at the beginning of the study. If  $k = 0.05$  find the size of the rabbit population after six years.
- (b) How long will it take the rabbit population to double in size?

5. Radium decays exponentially and its half-life is 1600 years.

If  $A_0$  represents the amount of radium in a sample to start with and  $A(t)$  represents the amount remaining after  $t$  years, then  $A(t) = A_0 e^{-kt}$ .

- (a) Determine the value of  $k$ , correct to 3 significant figures.
- (b) Hence find what percentage, to the nearest whole number, of the original amount of radium will be remaining after 2500 years.

6. The concentration of a fertiliser in the soil can be modelled by the equation  $F = F_0 e^{-kt}$  where  $F_0$  is the initial concentration,  $F_t$  is the concentration at time  $t$  and  $t$  is the time, in days, after the application of the pesticide.

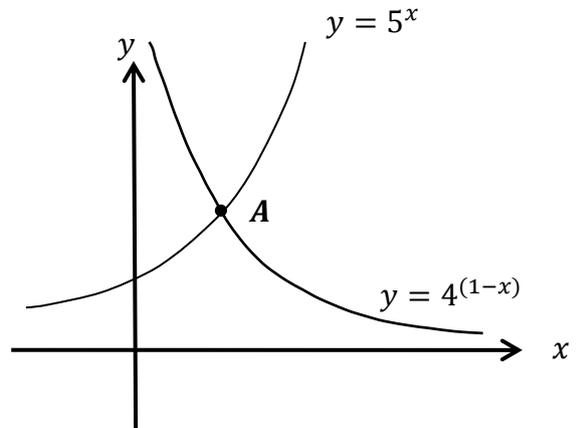
- (a) If it takes 20 days for the level of the fertiliser in the soil to reduce by 25%, find the value of  $k$  to 2 significant figures.
- (b) Eighty days after the initial application, what is the percentage decrease in the concentration of the fertiliser?

7. The spread of disease in trees was described by a law of the form  $A(t) = A_0 e^{kt}$  where  $A_0$  is the area covered by the disease when it was first detected and  $A$  is the area covered by the disease  $t$  months later.

If it takes six months for the area of the disease to double, find the value of the constant,  $k$ , correct to 3 significant figures.

**NR5** I can display on, and extract information from, logarithmic and exponential graphs.

1. The diagram shows the curves with equations  $y = 5^x$  and  $y = 4^{(1-x)}$ .



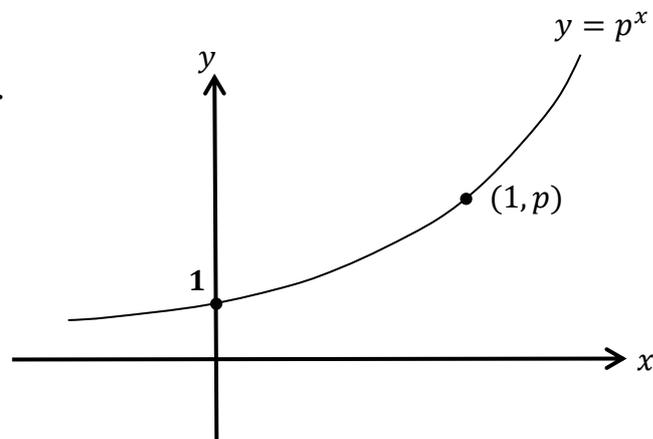
The graphs intersect at the point A.

- (a) Show that the x-coordinate of A can be written in the form  $\frac{\log_a C}{\log_a D}$ , for all  $a > 1$ .
- (b) Calculate the y-coordinate of A.

2. The diagram shows the graph of  $y = p^x$ ,  $p > 1$ .

On separate diagrams sketch:

- (a)  $y = -p^x$
- (b)  $y = p^{-x}$
- (c)  $y = p^{2-x}$



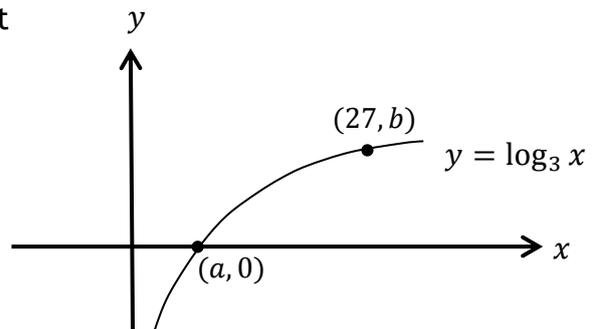
3. Sketch the graph of  $y = \log_a x$ . On the same diagram sketch:

- (a)  $y = \log_a \left(\frac{1}{x}\right)$
- (b)  $y = \log_a(x - 3)$
- (c)  $y = \log_a x + 2$

4. Sketch the graph of  $y = p^x + 1$ .
- (a) On the same diagram sketch the graph of  $y = p^{x+1}$ .
- (b) Prove that the two graphs intersect at a point where the x-coordinate is  $\log_p\left(\frac{1}{p-1}\right)$ .

5. The diagram shows a sketch of part of the graph of  $y = \log_3 x$

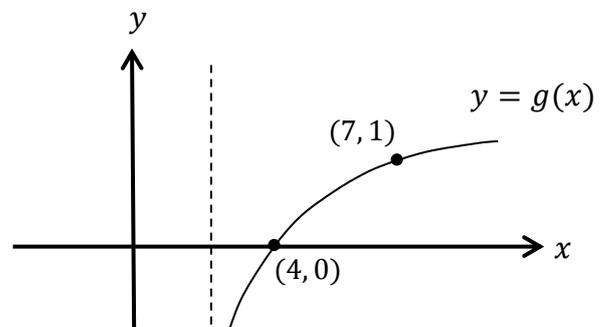
- (a) Write down the values of  $a$  and  $b$ .
- (b) Sketch the graph of  $y = \log_3(x - 1) + 2$



6. The function  $g$  is of the form  $g(x) = \log_p(x - q)$ .

The graph of  $y = g(x)$  is shown in the diagram.

- (a) Write down the values of  $p$  and  $q$ .
- (b) State the domain of  $g$ .



### Logarithms with composite functions

1. Functions  $p$ ,  $q$  and  $r$  are defined on suitable domains by

$$p(x) = x^2 - 12x + 19, q(x) = 5 - x \text{ and } r(x) = \log_2 x.$$

- (a) Find expressions for  $r(p(x))$  and  $r(q(x))$ .  
(b) Hence solve  $r(p(x)) - r(q(x)) = 3$ .

### Logarithms with polynomials

2. (a) Show that  $x = -4$  is a root of  $x^3 + 8x^2 + 11x - 20 = 0$ .  
Hence factorise  $x^3 + 8x^2 + 11x - 20$  fully.  
(b) Solve  $\log_2(x + 3) + \log_2(x^2 + 5x - 4) = 3$ .